

L^p - L^2 FOURIER RESTRICTION FOR HYPERSURFACES IN \mathbb{R}^3 : PART I

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ABSTRACT. This is the first of two articles, in which we prove a sharp L^p - L^2 Fourier restriction theorem for a large class of smooth, finite type hypersurfaces in \mathbb{R}^3 , which includes in particular all real-analytic hypersurfaces.

CONTENTS

1. Introduction	1
2. Preliminaries: Linear height, van der Corput type estimates	11
3. Normal forms of ϕ under linear coordinate changes when $h_{\text{lin}} < 2$	14
4. Reduction to restriction estimates near the principal root jet	17
5. The case when $h_{\text{lin}}(\phi) < 2$	22
5.1. The sub-case where $2^{2j}\delta_0 \gg 1$	24
5.2. The sub-case where $2^{2j}\delta_0 \ll 1$	28
5.3. The sub-case where $2^{2j}\delta_0 \sim 1$	30
6. Proof of Proposition 5.3: Airy type analysis	35
6.1. Estimation of $T_{\delta, Ai}^\lambda$	41
6.2. Estimation of $T_{\delta, l}^\lambda$	44
7. The case when $h_{\text{lin}}(\phi) \geq 2$: preparatory results	47
8. Restriction estimates in the transition domains when $h_{\text{lin}}(\phi) \geq 2$	48
9. Restriction estimates in the domains D_l , $l < l_{\text{pr}}$, when $h_{\text{lin}}(\phi) \geq 2$	55
10. Restriction estimates in the domain D_{pr} when $h_{\text{lin}}(\phi) \geq 5$	60
10.1. First step of the algorithm	61
10.2. Further steps of the algorithm	63
11. Necessary Conditions, and proof of Proposition 1.9	66
References	69

1. INTRODUCTION

Let S be a smooth, finite type hypersurface in \mathbb{R}^3 with Riemannian surface measure $d\sigma$, and consider the compactly supported measure $d\mu := \rho d\sigma$ on S , where $0 \leq \rho \in$

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$C_0^\infty(S)$. The goal of this article is to determine, possibly with the exception of the endpoint, the sharp range of exponents p for which a Fourier restriction estimate

$$(1.1) \quad \left(\int_S |\widehat{f}|^2 d\mu \right)^{1/2} \leq C_p \|f\|_{L^p(\mathbb{R}^3)}, \quad f \in \mathcal{S}(\mathbb{R}^3),$$

holds true. To this end, we may localize to a sufficiently small neighborhoods of a given point x^0 on S . Observe also that if estimate (1.1) holds for the hypersurface S , then it is valid also for every affine-linear image of S , possibly with a different constant if the Jacobian of this map is not one. By applying a suitable Euclidean motion of \mathbb{R}^3 we may then assume that $x^0 = (0, 0, 0)$, and that S is the graph

$$S = \{(x_1, x_2, \phi(x_1, x_2)) : (x_1, x_2) \in \Omega\},$$

of a smooth function ϕ defined on a sufficiently small neighborhood Ω of the origin, such that $\phi(0, 0) = 0$, $\nabla\phi(0, 0) = 0$.

In our preceding article [14], this problem had been solved, in terms of Newton diagrams associated to ϕ , under the assumption that there exists a linear coordinate system which is adapted to the function ϕ , in the sense of Varchenko. More precisely, if denote by $h(\phi)$ the height of ϕ , in the sense of Varchenko, then we had proved the following result:

Theorem 1.1. *Assume that, after applying a suitable linear change of coordinates, the coordinates (x_1, x_2) are adapted to ϕ . We then define the critical exponent p_c by*

$$(1.2) \quad p'_c := 2h(\phi) + 2,$$

where p' denotes the exponent conjugate to p , i.e., $1/p + 1/p' = 1$.

Then there exists a neighborhood $U \subset S$ of the point x^0 such that for every non-negative density $\rho \in C_0^\infty(U)$ the Fourier restriction estimate (1.1) holds true for every p such that

$$(1.3) \quad 1 \leq p \leq p_c.$$

Moreover, if $\rho(x^0) \neq 0$, then the condition (1.3) on p is also necessary for the validity of (1.1).

From now on, we shall therefore always make the following

Assumption 1.2. *There is no linear coordinate system which is adapted to ϕ .*

In order to formulate our main result, we need more notation. We shall build on the results and technics developed in [12] and [13], which will be our main references, also for references to earlier and related work. Let us first recall some basic notions from [12], which essentially go back to Arnol'd (cf. [2], [3]) and his school, most notably Varchenko [24].

If ϕ is given as before, consider the associated Taylor series

$$\phi(x_1, x_2) \sim \sum_{\alpha_1, \alpha_2=0}^{\infty} c_{\alpha_1, \alpha_2} x_1^{\alpha_1} x_2^{\alpha_2}$$

of ϕ centered at the origin. The set

$$\mathcal{T}(\phi) := \{(\alpha_1, \alpha_2) \in \mathbb{N}^2 : c_{\alpha_1, \alpha_2} = \frac{1}{\alpha_1! \alpha_2!} \partial_1^{\alpha_1} \partial_2^{\alpha_2} \phi(0, 0) \neq 0\}$$

will be called the *Taylor support* of ϕ at $(0, 0)$. We shall always assume that

$$\mathcal{T}(\phi) \neq \emptyset,$$

i.e., that the function ϕ is of finite type at the origin. The *Newton polyhedron* $\mathcal{N}(\phi)$ of ϕ at the origin is defined to be the convex hull of the union of all the quadrants $(\alpha_1, \alpha_2) + \mathbb{R}_+^2$ in \mathbb{R}^2 , with $(\alpha_1, \alpha_2) \in \mathcal{T}(\phi)$. The associated *Newton diagram* $\mathcal{N}_d(\phi)$ in the sense of Varchenko [24] is the union of all compact faces of the Newton polyhedron; here, by a *face*, we shall mean an edge or a vertex.

We shall use coordinates (t_1, t_2) for points in the plane containing the Newton polyhedron, in order to distinguish this plane from the (x_1, x_2) - plane.

The *Newton distance*, or shorter *distance* $d = d(\phi)$ between the Newton polyhedron and the origin in the sense of Varchenko is given by the coordinate d of the point (d, d) at which the bi-sectrix $t_1 = t_2$ intersects the boundary of the Newton polyhedron.

The *principal face* $\pi(\phi)$ of the Newton polyhedron of ϕ is the face of minimal dimension containing the point (d, d) . Deviating from the notation in [24], we shall call the series

$$\phi_{\text{pr}}(x_1, x_2) := \sum_{(\alpha_1, \alpha_2) \in \pi(\phi)} c_{\alpha_1, \alpha_2} x_1^{\alpha_1} x_2^{\alpha_2}$$

the *principal part* of ϕ . In case that $\pi(\phi)$ is compact, ϕ_{pr} is a mixed homogeneous polynomial; otherwise, we shall consider ϕ_{pr} as a formal power series.

Note that the distance between the Newton polyhedron and the origin depends on the chosen local coordinate system in which ϕ is expressed. By a *local coordinate system at the origin* we shall mean a smooth coordinate system defined near the origin which preserves 0. The *height* of the smooth function ϕ is defined by

$$h(\phi) := \sup\{d_y\},$$

where the supremum is taken over all local coordinate systems $y = (y_1, y_2)$ at the origin, and where d_y is the distance between the Newton polyhedron and the origin in the coordinates y .

A given coordinate system x is said to be *adapted* to ϕ if $h(\phi) = d_x$.

In [12] we proved that one can always find an adapted local coordinate system in two dimensions, thus generalizing the fundamental work by Varchenko [24] who worked in the setting of real-analytic functions ϕ (see also [17]).

Recall also that if the principal face of the Newton polyhedron $\mathcal{N}(\phi)$ is a compact edge, then it lies on a unique “principal line”

$$L := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1 t_1 + \kappa_2 t_2 = 1\},$$

with $\kappa_1, \kappa_2 > 0$. By permuting the coordinates x_1 and x_2 , if necessary, we shall always assume that $\kappa_1 \leq \kappa_2$. The weight $\kappa = (\kappa_1, \kappa_2)$ will be called the *principal weight* associated to ϕ . It induces dilations $\delta_r(x_1, x_2) := (r^{\kappa_1} x_1, r^{\kappa_2} x_2)$, $r > 0$, on \mathbb{R}^2 , so that

the principal part ϕ_{pr} of ϕ is κ -homogeneous of degree one with respect to these dilations, i.e., $\phi_{\text{pr}}(\delta_r(x_1, x_2)) = r\phi_{\text{pr}}(x_1, x_2)$ for every $r > 0$, and

$$(1.4) \quad d = \frac{1}{\kappa_1 + \kappa_2} = \frac{1}{|\kappa|}.$$

More generally, if $\kappa = (\kappa_1, \kappa_2)$ is any weight with $0 < \kappa_1 \leq \kappa_2$ such that the line $L_\kappa := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1 t_1 + \kappa_2 t_2 = 1\}$ is a supporting line to the Newton polyhedron $\mathcal{N}(\phi)$ of ϕ , then the κ -principal part of ϕ

$$\phi_\kappa(x_1, x_2) := \sum_{(\alpha_1, \alpha_2) \in L_\kappa} c_{\alpha_1, \alpha_2} x_1^{\alpha_1} x_2^{\alpha_2}$$

is a non-trivial polynomial which is κ -homogeneous of degree 1 with respect to the dilations associated to this weight as before. By definition, we then have

$$\phi(x_1, x_2) = \phi_\kappa(x_1, x_2) + \text{terms of higher } \kappa\text{-degree}$$

Adaptedness of a given coordinate system can be verified by means of the following criterion (see [12]): Denote by

$$m(\phi_{\text{pr}}) := \text{ord}_{S^1} \phi_{\text{pr}}$$

the maximal order of vanishing of ϕ_{pr} along the unit circle S^1 centered at the origin. The *homogeneous distance* of a κ -homogeneous polynomial P (such as $P = \phi_{\text{pr}}$) is given by $d_h(P) := 1/(\kappa_1 + \kappa_2) = 1/|\kappa|$. Notice that $(d_h(P), d_h(P))$ is just the point of intersection of the line given by $\kappa_1 t_1 + \kappa_2 t_2 = 1$ with the bi-sectrix $t_1 = t_2$. The height of P can be computed by means of the formula

$$(1.5) \quad h(P) = \max\{m(P), d_h(P)\}.$$

According to [12], Corollary 4.3 and Corollary 2.3, *the coordinates x are adapted to ϕ if and only if one of the following conditions is satisfied:*

- (a) *The principal face $\pi(\phi)$ of the Newton polyhedron is a compact edge, and $m(\phi_{\text{pr}}) \leq d(\phi)$.*
- (b) *$\pi(\phi)$ is a vertex.*
- (c) *$\pi(\phi)$ is an unbounded edge.*

We like to mention that in case (a) we have $h(\phi) = h(\phi_{\text{pr}}) = d_h(\phi_{\text{pr}})$. Notice also that (a) applies whenever $\pi(\phi)$ is a compact edge and $\kappa_2/\kappa_1 \notin \mathbb{N}$; in this case we even have $m(\phi_{\text{pr}}) < d(\phi)$ (cf. [12], Corollary 2.3).

In the case where the coordinates (x_1, x_2) are not adapted to ϕ , we see that the principal face $\pi(\phi)$ is a compact edge lying on a unique line

$$L = \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1 t_1 + \kappa_2 t_2 = 1\},$$

and that $m := \kappa_2/\kappa_1 \in \mathbb{N}$. Now, if $\kappa_2/\kappa_1 = 1$, then a linear change of coordinates of the form $y_1 = x_1, y_2 = x_2 - b_1 x_1$ will transform ϕ into a function $\tilde{\phi}$ for which, by our assumption, the coordinates (y_1, y_2) are still not adapted (cf. [12]). Replacing ϕ by $\tilde{\phi}$,

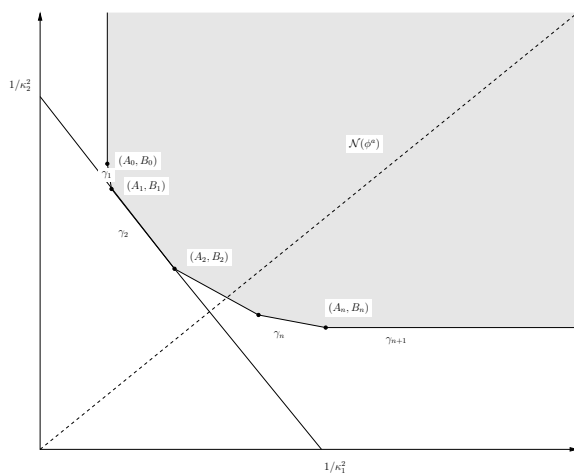


FIGURE 1. Edges and weights

it is also immediate that estimate (1.1) will hold for the graph of ϕ if and only if it holds for the graph of $\tilde{\phi}$. Replacing ϕ by $\tilde{\phi}$, we may and shall therefore always assume that our original coordinate system (x_1, x_2) is chosen so that

$$(1.6) \quad m = \kappa_2 / \kappa_1 \in \mathbb{N} \quad \text{and } m \geq 2.$$

Such a linear coordinate system will be called *linearly adapted* to ϕ (see Section 3 for a more comprehensive discussion of this notion).

Then, by Theorem 5.1 in [12], there exists a smooth real-valued function ψ (which we may choose as the so-called principal root jet of ϕ) of the form

$$(1.7) \quad \psi(x_1) = cx_1^m + O(x_1^{m+1})$$

with $c \neq 0$ defined on a neighborhood of the origin such that an adapted coordinate system (y_1, y_2) for ϕ is given locally near the origin by means of the (in general non-linear) shear

$$(1.8) \quad y_1 := x_1, \quad y_2 := x_2 - \psi(x_1).$$

In these coordinates, ϕ is given by

$$(1.9) \quad \phi^a(y) := \phi(y_1, y_2 + \psi(y_1)).$$

We remark that such an adapted coordinate system can be constructed by means of an algorithm which goes back Varchenko [24] in the case of real-analytic ϕ (see [12]).

Let us then denote the vertices of the Newton polyhedron $\mathcal{N}(\phi^a)$ by (A_l, B_l) , $l = 0, \dots, n$, where we assume that they are ordered so that $A_{l-1} < A_l$, $l = 1, \dots, n$, with associated compact edges given by the intervals $\gamma_l := [(A_{l-1}, B_{l-1}), (A_l, B_l)]$, $l = 1, \dots, n$. The unbounded horizontal edge with left endpoint (A_n, B_n) will be denoted by γ_{n+1} . To each of these edges γ_l , we associate the weight $\kappa^l = (\kappa_1^l, \kappa_2^l)$, so that γ_l is

contained in the line

$$L_l := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1^l t_1 + \kappa_2^l t_2 = 1\}.$$

For $l = n + 1$, we have $\kappa_1^{n+1} := 0, \kappa_2^{n+1} = 1/B_n$. We denote by

$$a_l := \frac{\kappa_2^l}{\kappa_1^l}, \quad l = 1, \dots, n$$

the reciprocal of the slope of the line L_l . For $l = n + 1$, we formally set $a_{n+1} := \infty$.

If $l \leq n$, the κ^l -principal part ϕ_{κ^l} of ϕ corresponding to the supporting line L_l is of the form

$$(1.10) \quad \phi_{\kappa^l}(x) = c_l x_1^{A_{l-1}} x_2^{B_l} \prod_{\alpha} \left(x_2 - c_l^{\alpha} x_1^{a_l} \right)^{N_{\alpha}}$$

(cf. [13]). In view of this identity, we shall say that the edge $\gamma_l := [(A_{l-1}, B_{l-1}), (A_l, B_l)]$ is associated to the cluster of roots $[l]$.

Consider the line parallel to the bi-sectrix

$$\Delta^{(m)} := \{(t, t + m + 1) : t \in \mathbb{R}\}.$$

For any edge $\gamma_l \subset L_l := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1^l t_1 + \kappa_2^l t_2 = 1\}$ define h_l by

$$\Delta^{(m)} \cap L_l = \{(h_l - m, h_l + 1)\},$$

i.e.,

$$(1.11) \quad h_l = \frac{1 + m\kappa_1^l - \kappa_2^l}{\kappa_1^l + \kappa_2^l},$$

and define the *restriction height*, or short, *r-height*, of ϕ by

$$h^r(\phi) := \max(d, \max_{\{l=1, \dots, n+1 : a_l > m\}} h_l).$$

Remarks 1.3. (a) For L in place of L_l and κ in place of κ^l , one has $m = \kappa_2/\kappa_1$ and $d = 1/(\kappa_1 + \kappa_2)$, so that one gets d in place of h_l in (1.11)

(b) Since $m < a_l$, we have $h_l < 1/(\kappa_1^l + \kappa_2^l)$, hence $h^r(\phi) < h(\phi)$.

It is easy to see by Remark 1.3 (a) that the *r-height* admits the following *geometric interpretation*:

By following Varchenko's algorithm (cf. Subsection 8.2 of [13]), one realizes that the Newton polyhedron of ϕ^a intersects the line L of the Newton polyhedron of ϕ in a compact face, either in a single vertex, or a compact edge. I.e., the intersection contains at least one and at most two vertices of ϕ^a , and we choose (A_{l_0-1}, B_{l_0-1}) as the one with smallest second coordinate. Then l_0 is the smallest index l such that γ_l has a slope smaller than the slope of L , i.e., $a_{l_0-1} \leq m < a_{l_0}$. We may thus consider the “augmented” Newton polyhedron $\mathcal{N}^r(\phi^a)$ of ϕ^a , which is the convex hull of the union of $\mathcal{N}(\phi^a)$ with the half-line $L^+ \subset L$ with right endpoint (A_{l_0-1}, B_{l_0-1}) . Then $h^r(\phi) + 1$ is the second coordinate of the point at which the line $\Delta^{(m)}$ intersects the boundary of $\mathcal{N}^r(\phi^a)$.

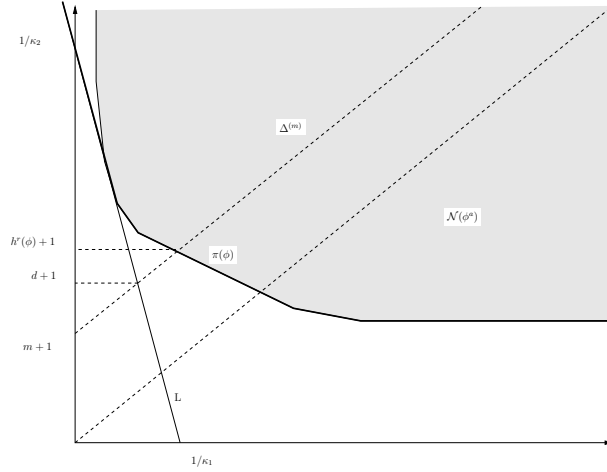


FIGURE 2. r-height

Theorem 1.4. *Let $\phi \neq 0$ be real analytic, and assume that there is no linear coordinate system adapted to ϕ . Then there exists a neighborhood $U \subset S$ of $x^0 = 0$ such that for every non-negative density $\rho \in C_0^\infty(U)$, the Fourier restriction estimate (1.1) holds true for every $p \geq 1$ such that $p' \geq p'_c := 2h^r(\phi) + 2$.*

- Remarks 1.5.**
- (a) *An application of Greenleaf's result would imply, at best, that the condition $p' \geq 2h(\phi) + 2$ is sufficient for (1.1) to hold, which is a strictly stronger condition than $p' \geq p'_c$.*
 - (b) *A. Seeger recently informed us that in a preprint, which regrettably had remained unpublished, Schulz [21] had already observed this kind of phenomenon for particular examples of surfaces of revolution.*
 - (c) *It can be shown that the number m is well-defined, i.e., it does not depend on the chosen linearly adapted coordinate system x (cf. Proposition 2.1).*

Example 1.6.

$$\phi(x_1, x_2) := (x_2 - x_1^m)^n, \quad n, m \geq 2.$$

The coordinates (x_1, x_2) are not adapted. Adapted coordinates are $y_1 := x_1, y_2 := x_2 - x_1^m$, in which ϕ is given by

$$\phi^a(y_1, y_2) = y_2^n.$$

Here

$$\kappa_1 = \frac{1}{mn}, \quad \kappa_2 = \frac{1}{n},$$

$$d := d(\phi) = \frac{1}{\kappa_1 + \kappa_2} = \frac{nm}{m+1},$$

and

$$p'_c = \begin{cases} 2d + 2, & \text{if } n \leq m + 1, \\ 2n, & \text{if } n > m + 1. \end{cases}$$

On the other hand, $h := h(\phi) = n$, so that $2h + 2 = 2n + 2 > p'_c$.

An analogous theorem holds true even for smooth, finite type functions ϕ , under an additional condition which, however, is always satisfied when ϕ is real-analytic. To state this more general result, and in order to prepare a more invariant description of the notion of r -height, we need to introduce more notation. Again, we shall assume that the coordinates (x_1, x_2) are linearly adapted to ϕ .

Definitions. Denote by $\mathbb{R}_\pm := \{x_1 \in \mathbb{R} : \pm x_1 > 0\}$ and by $H^\pm := \mathbb{R} \times \mathbb{R}_\pm$ the corresponding right, respectively left half-plane.

We say that a function $f = f(x_1)$ defined in $U \cap \mathbb{R}_+$ (respectively $U \cap \mathbb{R}_-$), where U is an open neighborhood of the origin, is *fractionally smooth*, if there exist a smooth function g on U and a positive integer q such that $f(x_1) = g(|x_1|^{1/q})$ for $x_1 \in U \cap \mathbb{R}_+$ (respectively $x_1 \in U \cap \mathbb{R}_-$). Moreover, we shall say that a fractionally smooth function f is *flat*, if $f(x_1) = O(|x_1|^N)$ for every $N \in \mathbb{N}$. Two smooth functions f and g defined on a neighborhood of the origin will be called *equivalent*, and we shall write $f \sim g$, if $f - g$ is flat. Finally, a *fractional shear* in H^\pm will be a change of coordinates of the form

$$y_1 := x_1, \quad y_2 := x_2 - f(x_1),$$

where f is real-valued and fractionally smooth, but not flat. If we express the smooth function ϕ on, say, the half-plane H^+ , as a function of $y = (y_1, y_2)$, the resulting function $\phi^f(y) = \phi(y_1, y_2 + f(y_1))$ will in general no longer be smooth at the origin, but “fractionally smooth”.

For such functions, there are straight-forward generalizations of the notions of Newton-polyhedron, etc.. Namely, following [13], and assuming without loss of generality that we are in H^+ where $x_1 > 0$, let ϕ be a function of the variables $x_1^{1/q}$ and x_2 near the origin, i.e., there exists a smooth function $\phi^{[q]}$ near the origin such that $\phi(x) = \phi^{[q]}(x_1^{1/q}, x_2)$ (more generally, we could assume that ϕ is a smooth function of the variables $x_1^{1/q}$ and $x_2^{1/p}$, where p and q are positive integers, but we won't need this generality here). Such functions ϕ will also be called *fractionally smooth*. If the Taylor series of $\phi^{[q]}$ is given by

$$\phi^{[q]}(x_1, x_2) \sim \sum_{\alpha_1, \alpha_2=0}^{\infty} c_{\alpha_1, \alpha_2} x_1^{\alpha_1} x_2^{\alpha_2},$$

then ϕ has the formal Puiseux series expansion

$$\phi(x_1, x_2) \sim \sum_{\alpha_1, \alpha_2=0}^{\infty} c_{\alpha_1, \alpha_2} x_1^{\alpha_1/q} x_2^{\alpha_2}.$$

We therefore define the *Taylor-Puiseux support*, or shorter, *Taylor-support* of ϕ by

$$\mathcal{T}(\phi) := \{(\frac{\alpha_1}{q}, \alpha_2) \in \mathbb{N}_q^2 : c_{\alpha_1, \alpha_2} \neq 0\},$$

where $\mathbb{N}_q^2 := (\frac{1}{q}\mathbb{N}) \times \mathbb{N}$. The *Newton-Puiseux polyhedron* (shorter: *Newton polyhedron*) $\mathcal{N}(\phi)$ of ϕ at the origin is then defined to be the convex hull of the union of all the

quadrants $(\alpha_1/q, \alpha_2) + \mathbb{R}_+^2$ in \mathbb{R}^2 , with $(\alpha_1/q, \alpha_2) \in \mathcal{T}(\phi)$, and other notions, such as the notion of principal face, Newton distance or homogenous distance, are defined in analogy with our previous definitions for smooth functions ϕ .

Now, if $f(x_1)$ has the formal Puiseux series expansion (say for $x_1 > 0$)

$$f(x_1) \sim \sum_{j \geq 0} c_j x_1^{m_j},$$

with non-zero coefficients c_j and exponents m_j which are growing with j and are all multiples of $1/q$, we isolate the leading exponent m_0 and choose the weight κ^f so that $\kappa_2^f / \kappa_1^f = m_0$ and such that the line

$$L^f := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1^f t_1 + \kappa_2^f t_2 = 1\}$$

is a supporting line to $\mathcal{N}(\phi^f)$. We can then define the augmented Newton polyhedron $\mathcal{N}^r(\phi^f)$ in the same way as we defined $\mathcal{N}^r(\phi^a)$, replacing the exponent m by m_0 and the line L by L^f , and define, in analogy with $h^r(\phi)$, the r -height $h^f(\phi)$ associated to f by requiring that $h^f(\phi) + 1$ is the second coordinate of the point at which the line $\Delta^{(m_0)}$ intersects the boundary of $\mathcal{N}^r(\phi^f)$. Again, it is easy to see that

$$(1.12) \quad h^f(\phi) = \max(d^f, \max_{\{l: a_l > m_0\}} h_l^f),$$

where (d^f, d^f) is the point of intersection of the line L^f with the bi-sectrix, and where h_l^f is associated to the edge γ_l of $\mathcal{N}(\phi^f)$ by the analogue of formula (1.11), i.e.,

$$(1.13) \quad h_l^f = \frac{1 + m_0 \kappa_1^l - \kappa_2^l}{\kappa_1^l + \kappa_2^l},$$

if γ_l is again contained in the line L_l defined by the weight κ^l .

Finally, let us say that a fractionally smooth function $f(x_1)$ *agrees with the principal root jet* $\psi(x_1)$ *up to terms of higher order*, if the following holds: if ψ is not a polynomial, then $f \sim \psi$, and if ψ is polynomial of degree D , then the leading exponent in the formal Puiseux expansion of $f - \psi$ is strictly bigger than D .

We can now formulate the condition that we need when ϕ is non-analytic.

Condition (R). For every fractionally smooth, real function $f(x_1)$ which agrees with the principal root jet $\psi(x_1)$ up to terms of higher order, the following holds true:

If $B \in \mathbb{N}$ is maximal such that $\mathcal{N}(\phi^f) \subset \{(t_1, t_2) : t_2 \geq B\}$, then ϕ factors as $\phi(x_1, x_2) = (x_2 - \tilde{f}(x_1))^B \tilde{\phi}(x_1, x_2)$, where $\tilde{f} \sim f$ and where $\tilde{\phi}$ is fractionally smooth.

Clearly, Condition (R) is satisfied if ϕ is real-analytic.

Theorem 1.7. *Let ϕ be smooth and of finite type, and assume that the coordinates (x_1, x_2) are linearly adapted to ϕ , but not adapted, and that Condition (R) is satisfied.*

Then there exists a neighborhood $U \subset S$ of $x^0 = 0$ such that for every non-negative density $\rho \in C_0^\infty(U)$, the Fourier restriction estimate (1.1) holds true for every $p \geq 1$ such that $p' \geq p'_c := 2h^r(\phi) + 2$.

This theorem is sharp in the following sense:

Theorem 1.8. *Let ϕ be smooth of finite type, and assume that the Fourier restriction estimate (1.1) holds true in a neighborhood of x^0 . Then, if $\rho(x^0) \neq 0$, necessarily $p' \geq p'_c$.*

Finally, we can also give a more invariant description of the notion of r -height, which conceptually resembles more closely Varchenko's definition of the notion of height, only that we restrict the admissible changes of coordinates to the class of fractional shears in the half-planes H^+ and H^- . Assume again that the coordinates (x_1, x_2) are linearly adapted to ϕ , and let

$$(1.14) \quad \tilde{h}^r(\phi) := \sup_f h^f(\phi),$$

where the supremum is taken over all non-flat fractionally smooth, real functions $f(x_1)$ of $x_1 > 0$ (corresponding to a fractional shear in H^+) or of $x_1 < 0$ (corresponding to a fractional shear in H^-). Then obviously

$$(1.15) \quad h^r(\phi) \leq \tilde{h}^r(\phi),$$

but in fact there is equality:

Proposition 1.9. *Assume that the coordinates (x_1, x_2) are linearly adapted to ϕ , where ϕ is smooth and of finite type and satisfies $\phi(0, 0) = 0$, $\nabla\phi(0, 0) = 0$.*

- (a) *If the coordinates (x_1, x_2) are not adapted to ϕ , then for every non-flat fractionally smooth, real function $f(x_1)$ and the corresponding fractional shear in H^+ respectively H^- , we have $h^f(\phi) \leq h^r(\phi)$. Consequently, $h^r(\phi) = \tilde{h}^r(\phi)$.*
- (b) *If the coordinates (x_1, x_2) are adapted to ϕ , then $\tilde{h}^r(\phi) = d(\phi) = h(\phi)$.*

In particular, the critical exponent for the restriction estimate (1.1) is in all cases given by $p'_c := 2\tilde{h}^r(\phi) + 2$.

Organization of the article: Before we turn to the proof of Theorem 1.7, we shall first clarify the notion of linearly adapted coordinates in Section 2.

Moreover, as in the preceding papers [13], [14], assuming that the coordinates x are linearly adapted, it will be natural to distinguish the cases where $d(\phi) < 2$ and where $d(\phi) \geq 2$, since, in contrast to the first case, in the latter case in many situations a reduction to estimates for one-dimensional oscillatory integrals will be possible, which in return can be performed by means of van der Corput's lemma ([22]), respectively the van der Corput type Lemma 2.2. The latter result will be stated too in Section 2.

Our discussion of the case where $d(\phi) < 2$ will rely on certain normal forms to which ϕ can be transformed by means of a linear change of coordinates. These will be derived in Section 3.

Next, in Section 4, as a first step in the proof of Theorem 1.7 we shall show that one may reduce the restriction estimate to the piece of surface which lies above a small

“conic ” neighborhood of the principal root jet ψ . This step works in all cases, no matter what the value of $d(\phi)$ is.

Sections 5 and 6 will be devoted to the proof of Theorem 1.7 in the case where $d(\phi) < 2$. Some of the main tools will consist of various kinds of dyadic domain decompositions in combination with Littlewood-Paley theory and re-scaling arguments, and additional dyadic decompositions in frequency space. It turns out that the particular case where $m = 2$ in (1.6), (1.7) requires a more refined analysis than the case $m \geq 3$. Indeed, in this case, it turns out that further dyadic decompositions with respect to the distance to a certain “Airy cone” are needed. This particular case will be discussed in Section 6.

Sections 7 - 10 will deal with the case where $d(\phi) \geq 2$. It is natural to decompose the surface S according to the “root structure” of the function ϕ , which in return is reflected by properties of the Newton diagram associated to ϕ^a (cf. [16], [13] and [14]. More precisely, we shall decompose the domain Ω into certain domains D_l , which are homogeneous in adapted coordinates, and intermediate “transition” domains E_l , and consider the corresponding decomposition of the surface S . The particular domain D_l which contains the principal root jet $x_2 = \psi(x_1)$ will be called D_{pr} . It is this domain whose discussion will require the most refined arguments. All this is described in Section 7. Next, in Section 8, we estimate the contribution of the transition domains E_l to the restriction problem. It turns out that this works whenever $d(\phi) \geq 2$. Similarly, in Section 9 we can also treat the contributions by the domains D_l different from D_{pr} whenever $d(\phi) \geq 2$.

What remains is the domain D_{pr} . The contribution by this domain is studied in Section 10, by means of a certain domain decomposition algorithm, which, roughly speaking, reflects the “fine splitting” of roots of $\partial_2 \phi^a$. In this discussion, various cases arise, and there is one case in which we may fibre the corresponding piece of surface into a family of curves with non-vanishing torsion, so that we can apply Drury’s restriction theorem for curves [7]. However, it turns out that this requires that $d(\phi) \geq 5$.

What remains open at this stage is the proof of Proposition 4.3 in the case where $2 \leq h_{\text{lin}}(\phi) < 5$. The discussion of this case requires substantially more refined techniques and interpolation arguments, and will be the content of [15].

Finally, in Section 11, we shall employ a Knapp-type argument in order to show that the condition $p' \geq p'_c$ is necessary in Theorem 1.7, and conclude the article with a proof of Proposition 1.9.

2. PRELIMINARIES: LINEAR HEIGHT, VAN DER CORPUT TYPE ESTIMATES

In analogy with Varchenko’s notion of height, let us introduce the notion of *linear height* of ϕ , which measures the upper limit of all Newton distances of ϕ in linear coordinate systems:

$$h_{\text{lin}}(\phi) := \sup\{d(\phi \circ T) : T \in GL(2, \mathbb{R})\}.$$

Note that

$$d(\phi) \leq h_{\text{lin}}(\phi) \leq h(\phi).$$

We also say that a linear coordinate system $y = (y_1, y_2)$ is *linearly adapted* to ϕ , if $d_y = h_{\text{lin}}(\phi)$. Clearly, if there is a linear coordinate system which is adapted to ϕ , it is in particular linearly adapted to ϕ . The following proposition gives a characterization of linearly adapted coordinates under the complementary Assumption 1.2.

Proposition 2.1. *If ϕ satisfies Assumption 1.2, and if $\phi = \phi(x)$, then the following are equivalent:*

- (a) *The coordinates x are linearly adapted to ϕ .*
- (b) *If the principal face $\pi(\phi)$ is contained in the line*

$$L = \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1 t_1 + \kappa_2 t_2 = 1\},$$

then either $\kappa_2/\kappa_1 \geq 2$ or $\kappa_1/\kappa_2 \geq 2$.

Moreover, in all linearly adapted coordinates x for which $\kappa_2/\kappa_1 > 1$, the principal face of the Newton polyhedron is the same, so that in particular the number $m := \kappa_2/\kappa_1$ does not depend on the choice of the linearly adapted coordinate system.

This result shows in particular that linearly adapted coordinates always exist under Assumption 1.2, since either the original coordinates for ϕ are already linearly adapted, or we arrive at such coordinates after applying the first step in Varchenko's algorithm (when $\kappa_2/\kappa_1 = 1$ in the original coordinates).

Proof. In order to prove that (a) implies (b), assume that $d_x := d(\phi) = h_{\text{lin}}(\phi)$. By interchanging the coordinates x_1 and x_2 , if necessary, we may assume that $\kappa_2/\kappa_1 \geq 1$, where we recall that $\kappa_2/\kappa_1 \in \mathbb{N}$. Now, if we had $\kappa_2/\kappa_1 = 1$, then, by Varchenko's algorithm, there would exist a linear change of coordinates of the form $y_1 = x_1, y_2 = x_2 - cx_1$ so that $d_y > d_x = d$, which would contradict the maximality of d_x . Thus, necessarily $\kappa_2/\kappa_1 \geq 2$.

Conversely, assume without loss of generality that $\kappa_2/\kappa_1 \geq 2$. Consider any matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{R})$, and the corresponding linear coordinates y given by

$$x_1 = ay_1 + by_2, \quad x_2 = cy_1 + dy_2.$$

To prove (a), we have to show that $d_y \leq d_x$ for all such matrices T .

1. Case. $a \neq 0$. Then we may factorize $T = T_1 T_2$, where

$$T_1 := \begin{pmatrix} a & 0 \\ c & \frac{ad-bc}{a} \end{pmatrix}, \quad T_2 := \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}.$$

We first consider T_2 . Since $\phi_{\text{pr}}(T_2 y) = \phi_{\kappa}(y_1 + \frac{b}{a}y_2, y_2)$, where y_2 is κ -homogenous of degree $\kappa_2 > \kappa_1$, where κ_1 is the κ -degree of y_1 , we see that the κ -principal part of $\phi \circ T_2$ is given by $(\phi \circ T_2)_{\kappa} = \phi_{\kappa}$, so that $\phi \circ T_2$ and ϕ have the same principal face, and in particular the same Newton distance. This shows that we may assume without loss of generality that $b = 0$. Then necessarily $d \neq 0$. But then our change of coordinates is

of the type $x_1 = ay_1$, $x_2 = cy_1 + dy_2$ considered in Lemma 3.2 of [12], so that this lemma implies that $d_y \leq d_x$. Indeed, one finds more precisely that $d_y < d_x$, if $c \neq 0$, and $d_y = d_x$ otherwise.

2. Case. $a = 0, d = 0$. Since separate scalings of the coordinates have no effect on the Newton polyhedra, T then essentially interchanges the roles of x_1 and x_2 , i.e., the Newton polyhedron is reflected at the bi-sectrix under this coordinate change. This shows that here $d_y = d_x$.

3. Case. $a = 0, d \neq 0$. Then we may factorize $T = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} = T_1 T_2$, where

$$T_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_2 := \begin{pmatrix} c & d \\ 0 & b \end{pmatrix}.$$

We have seen in the previous cases that both T_1 and T_2 do not change the Newton distance, and thus here $d_y = d_x$. This concludes the proof of the first part of Proposition 2.1.

Assume finally that x and y are two linearly adapted coordinate systems for ϕ , for which the corresponding principal weights κ and κ' satisfy $\kappa_2/\kappa_1 > 1$ and $\kappa'_2/\kappa'_1 > 1$, respectively. Choose $T \in GL(2, \mathbb{R})$ such that $x = Ty$.

Inspecting the three cases from the previous argument, we see that in Case 1 the mapping T_2 does not change the principal face, and that necessarily $c = 0$, since otherwise we had $d_y < d_x$. But then also T_1 does not change the principal face. Case 2 cannot arise here, since we assume that both $\kappa_2/\kappa_1 > 1$ and $\kappa'_2/\kappa'_1 > 1$, and similarly Case 3 cannot apply. This proves also the second statement in the proposition.

Q.E.D.

We recall the following “van der Corput type lemma”, which is a (not completely straight-forward) consequence of the classical van der Corput lemma (see, e.g., [22]) and whose formulation goes back to J. E. Björk (see [6]) and G. I. Arhipov [1]).

Lemma 2.2. *Assume that f is a smooth real valued function defined on an interval $I \subset \mathbb{R}$ which is of polynomial type $M \geq 2$ ($M \in \mathbb{N}$), i.e., there are positive constants $c_1, c_2 > 0$ such that*

$$c_1 \leq \sum_{j=1}^M |f^{(j)}(s)| \leq c_2 \quad \text{for every } s \in I.$$

Then for $\lambda \in \mathbb{R}$,

$$\left| \int_I e^{i\lambda f(s)} g(s) ds \right| \leq C(\|g\|_{L^\infty(I)} + \|g'\|_{L^1(I)})(1 + |\lambda|)^{-1/M},$$

where the constant C depends only on the constants c_1 and c_2 .

3. NORMAL FORMS OF ϕ UNDER LINEAR COORDINATE CHANGES WHEN $h_{\text{lin}} < 2$

In this section we shall provide normal forms of the functions ϕ under linear coordinate changes when $h_{\text{lin}} < 2$. This extends Siersma's work on analytic functions [19] to the smooth, finite type case. The designation of the type of singularity that we list below corresponds to Arnol'd's classification of singularities in the case of analytic functions (cf. [3] and [8]), i.e., in the analytic case, non-linear analytic changes of coordinates would allow to further reduce ϕ to Arnol'd's normal forms.

Proposition 3.1. *Assume that $h_{\text{lin}}(\phi) < 2$, where ϕ satisfies Assumption 1.2.*

Then, after applying a suitable linear change of coordinates, ϕ can be written in the following form on a sufficiently small neighborhood of the origin:

$$(3.1) \quad \phi(x_1, x_2) = b(x_1, x_2)(x_2 - \psi(x_1))^2 + b_0(x_1),$$

where b, b_0 and ψ are smooth functions, and where $\psi(x_1) = cx_1^m + O(x_1^{m+1})$, with $c \neq 0$ and $m \geq 2$. Moreover, we can distinguish two cases:

Case a. $b(0, 0) \neq 0$. Then either

(i) b_0 is flat, (singularity of type A_∞)

or

(ii) $b_0(x_1) = x_1^n \beta(x_1)$, where $\beta(0) \neq 0$ and $n \geq 2m + 1$. (singularity of type A_{n-1})

In these cases we say that ϕ is of type A .

Case b. $b(0, 0) = 0$. Then we may assume that

$$(3.2) \quad b(x_1, x_2) = x_1 b_1(x_1, x_2) + x_2^2 b_2(x_2),$$

where b_1 and b_2 are smooth functions, with $b_1(0, 0) \neq 0$.

Moreover, either

(i) b_0 is flat, (singularity of type D_∞)

or

(ii) $b_0(x_1) = x_1^n \beta(x_1)$, where $\beta(0) \neq 0$ and $n \geq 2m + 2$. (singularity of type D_{n+1})

In these cases we say that ϕ is of type D .

Remarks 3.2. (a) It is easy to see that the Newton distance $d = d(\phi)$ for these normal forms is given as follows:

$$d = \begin{cases} \frac{2m}{m+1}, & \text{if } \phi \text{ is of type } A, \\ \frac{2m+1}{m+1}, & \text{if } \phi \text{ is of type } D, \end{cases}$$

and by Proposition 2.1 that $h_{\text{lin}}(\phi) = d$, i.e., that the coordinates x are linearly adapted.

(b) Similarly, the coordinates $y_1 := x_1$, $y_2 := x_2 - \psi(x_1)$ are adapted to ϕ , and we can choose ψ as the principal root jet.

- (c) When ϕ has a singularity of type A_∞ or D_∞ and satisfies Condition (R), then, after replacing ψ by an equivalent function, we may assume that $b_0 \equiv 0$.

Proof. If $D^2\phi(0,0)$ had full rank 2, then the coordinates x would already be adapted to ϕ , which would contradict our assumptions. Therefore $\text{rank } D^2\phi(0,0) \leq 1$. Let us denote by P_n the homogeneous part of degree n of the Taylor polynomial of ϕ , i.e., $P_n(x_1, x_2) = \sum_{j+k=n} c_{jk} x_1^j x_2^k$.

1. Case: $\text{rank } D^2\phi(0,0) = 1$.

In this case, by passing to a suitable linear coordinate system, we may assume that $P_2(x_1, x_2) = ax_2^2$, where $a \neq 0$. Consider the equation

$$\partial_2\phi(x_1, x_2) = 0.$$

By the implicit function theorem, it has locally a unique smooth solution $x_2 = \psi(x_1)$, i.e., $\partial_2\phi(x_1, \psi(x_1)) = 0$. A Taylor series expansion of the function $\phi(x_1, x_2)$ with respect to the variable x_2 around $\psi(x_1)$ then shows that

$$(3.3) \quad \phi(x_1, x_2) = b(x_1, x_2)(x_2 - \psi(x_1))^2 + b_0(x_1),$$

where b and b_0 are smooth functions and $b(0,0) = \frac{1}{2}\partial_2^2\phi(0,0) = a \neq 0$, whereas $b_0(x_1) = O(x_1^2)$, since $\phi(0,0) = 0$, $\nabla\phi(0,0) = 0$ (this is a special instance of what would follow from a classical division theorem, see, e.g., [11]).

Now, either b_0 is flat, which leads to type A_∞ , or otherwise we may write $b_0(x_1) = x_1^n\beta(x_1)$, where $\beta(0) \neq 0$ and $n \geq 2$, which leads to type A_{n-1} .

Observe also that the function ψ cannot be flat, for otherwise the Newton polyhedron of ϕ would be the set $(0,2) + \mathbb{R}_+^2$, in case that b_0 is flat, or its principal edge would be the compact line segment with vertices $(0,2)$ and $(n,0)$. In the latter case, the principal part of ϕ is given by $\phi_{\text{pr}}(x_1, x_2) = ax_2^2 + g(0)x_1^n$, so that the maximal multiplicity $m(\phi_{\text{pr}})$ of any real root of ϕ_{pr} along the unit circle is at most 1, whereas the Newton distance is given by $d = 1/(\frac{1}{2} + \frac{1}{n}) \geq 1$. Therefore, in both cases, the coordinates x would already be adapted to ϕ , according to Corollary 4.3 in [12]. Notice also that the same argument shows that the coordinates y introduced in (1.8) are adapted to ϕ , so that in particular indeed $h = 2$ (in case that b_0 is flat) respectively $h = 1/(\frac{1}{2} + \frac{1}{n}) < 2$ (if $b_0(x_1) = x_1^n\beta(x_1)$).

In particular, since $\psi(0) = 0$, we can write $\psi(x_1) = cx_1^m + O(x_1^{m+1})$ for some $m \in \mathbb{N}$, where $c \neq 0$. Note that indeed $m \geq 2$, since $P_2(x_1, x_2) = ax_2^2$.

Finally, when $b_0(x_1) = x_1^n\beta(x_1)$, a similar reasoning as before shows that the coordinates x are already adapted if $2m \geq n$, so that under Assumption 1.2 we must have $n \geq 2m + 1$.

2. Case: $D^2\phi(0,0) = 0$.

Then $P_2 = 0$, and $P_3 \neq 0$, for otherwise we had $h_{\text{lin}} \geq d \geq 1/(1/4 + 1/4) = 2$, which would contradict our assumption that $h_{\text{lin}} < 2$. Notice also that $P_3 \neq 0$ is homogeneous of odd degree 3, so that necessarily $m(P_3) \geq 1$.

Assume first that $m(P_3) = 1$. Then, passing to a suitable linear coordinate system, we may assume that $P_3(x_1, x_2) = x_1(x_2 - \alpha x_1)(x_2 - \beta x_1)$, where either $\alpha \neq \beta$ are both real, or $\alpha = \bar{\beta}$ are non-real. Then one checks easily that the Newton diagram of P_3 is a compact edge intersecting the bi-sectrix in its interior and contained in the line given by $\frac{1}{3}t_1 + \frac{1}{3}t_2 = 1$. Consequently, it agrees with the principal face $\pi(\phi)$, so that $P_3 = \phi_{\text{pr}}$. We thus find that the Newton distance d in this linear coordinate system satisfies $d = 3/2 > m(\phi_{\text{pr}})$, so that these coordinates would already be adapted, contradicting our assumptions.

Assume next that $m(P_3) = 3$. Then, in a suitable linear coordinate system, $P_3(x_1, x_2) = x_2^3$. These coordinates are then adapted to P_3 , so that $h(P_3) = d(P_3) = 3 > 2$. However, as has been shown in [13], p. 217, under Assumption 1.2 this implies that the Taylor support of ϕ is contained in the region where $\frac{1}{6}t_1 + \frac{1}{3}t_2 \geq 1$. This in return implies that $h_{\text{lin}} \geq d \geq 1/(\frac{1}{6} + \frac{1}{3}) = 2$, in contrast to what we assumed.

We have thus seen that necessarily $m(P_3) = 2$. Then, after applying a suitable linear change of coordinates, we may assume that $P_3(x_1, x_2) = x_1x_2^2$, i.e.,

$$\phi(x_1, x_2) = x_1x_2^2 + O(|x|^4).$$

Consider here the equation

$$(3.4) \quad \partial_1 \partial_2 \phi(x_1, x_2) = 0.$$

By the implicit function theorem, it has locally a unique smooth solution $x_2 = \psi(x_1)$, i.e., $\partial_1 \partial_2 \phi(x_1, \psi(x_1)) = 0$. By means of a Taylor series expansion of the function $\partial_1 \phi(x_1, x_2)$ with respect to the variable x_2 around $\psi(x_1)$ and subsequent integration in x_1 one then finds that

$$\phi(x_1, x_2) = b(x_1, x_2)(x_2 - \psi(x_1))^2 + b_2(x_1)x_2 + b_0(x_1),$$

where b, b_0 and b_2 are smooth functions. Again, we have that $\psi(x_1) = cx_1^m + O(x_1^{m+1})$, with $m \geq 2$. Then (3.4) implies that $b_2' = 0$, and since $\partial_2 \phi(0, 0) = 0$, we see that $b_2 = 0$, hence

$$(3.5) \quad \phi(x_1, x_2) = b(x_1, x_2)(x_2 - \psi(x_1))^2 + b_0(x_1),$$

Moreover, since $\partial_2^2 \phi(0, 0) = 0$, $\partial_1 \partial_2^2 \phi(0, 0) \neq 0$, $\partial_2^3 \phi(0, 0) = 0$, we have that

$$b(0, 0) = 0, \quad \partial_1 b(0, 0) \neq 0 \quad \text{and} \quad \partial_2 b(0, 0) = 0.$$

By Taylor's formula, this implies that

$$b(x_1, x_2) = x_1 b_1(x_1, x_2) + x_2^2 b_2(x_2),$$

where b_1 and b_2 are smooth functions, with $b_1(0, 0) \neq 0$.

In a similar way as in Case 1, one can see that the coordinates from (1.8) are adapted to ϕ . Moreover, if b_0 is flat, which leads to case D_∞ , then $h = 2$, and if $b_0(x_1) = x_1^n \beta(x_1)$, which leads to case D_{n+1} , then $h = \frac{2n}{n+1} < 2$. Finally, one also checks easily that the coordinates x in (1.8) are already adapted to ϕ , if $2m + 1 \geq n$, so that under our assumption we must have $n \geq 2m + 2$.

This concludes the proof of Proposition 3.1.

Q.E.D.

Corollary 3.3. *Assume that ϕ satisfies Assumption 1.2. By passing to a suitable linear coordinate system, let us also assume that the coordinates x are linearly adapted to ϕ . Then, if $d = d(\phi) < 2$, the critical exponent in Theorem 1.7 is given by $p'_c = 2d + 2$.*

Proof. Proposition 3.1 shows that the principal face $\pi(\phi)$ of the Newton polyhedron of ϕ is a compact edge whose “upper” vertex v is one of the following points $(0, 2)$ or $(1, 2)$, which both lie below the line $H := \{(t_1, t_2) : t_2 = 3\}$ within the positive quadrant. On the other hand, $m + 1 \geq 3$. It is then clear from the geometry of the lines H , the line L which contains $\pi(\phi)$ and the line $\Delta^{(m)}$, that $\Delta^{(m)}$ will intersect L above or in the vertex v . Since, by Varchenko’s algorithm, the point v will also be a vertex of the Newton polyhedron of ϕ^a , this easily implies that $h^r(\phi) = d$ (compare Figure 2). This proves the claim. Q.E.D.

4. REDUCTION TO RESTRICTION ESTIMATES NEAR THE PRINCIPAL ROOT JET

We now turn to the proof of Theorem 1.7 (which includes Theorem 1.4). As a first step, we shall reduce considerations to a small neighborhood of the principal root jet ψ . Recall that our coordinates x are assumed to satisfy (1.7) and (1.6).

Following [14], by decomposing \mathbb{R}^2 into its four quadrants, we shall in the sequel always assume that the surface carried measure $d\mu = \rho d\sigma$ is supported in the positive quadrant where $x_1 > 0, x_2 > 0$, i.e., that it is of the form

$$\langle \mu, f \rangle = \int_{(\mathbb{R}_+)^2} f(x, \phi(x)) \eta(x) dx, \quad f \in C_0(\mathbb{R}^3),$$

where $\eta(x) := \rho(x, \phi(x)) \sqrt{1 + |\nabla \phi(x)|^2}$ is smooth and has its support in the neighborhood Ω of the origin, which we may assume to be sufficiently small. The contributions by the other quadrants can be treated in a very similar way.

If χ is an integrable function defined on Ω , we put

$$\mu^\chi := (\chi \otimes 1)\mu, \quad \text{i.e., } \langle \mu^\chi, f \rangle = \int_{(\mathbb{R}_+)^2} f(x, \phi(x)) \eta(x) \chi(x) dx.$$

We choose a non-negative bump function $\chi_0 \in C_0^\infty(\mathbb{R})$ supported in $[-1, 1]$, and put

$$\rho_1(x_1, x_2) := \chi_0\left(\frac{x_2 - cx_1^m}{\varepsilon x_1^m}\right),$$

where $\varepsilon > 0$ is a small parameter to be determined later. Notice that ρ_1 is supported in the κ -homogeneous subdomain of $\Omega \cap \mathbb{R}^2$ where

$$(4.1) \quad |x_2 - cx_1^m| \leq \varepsilon x_1^m,$$

which contains the curve $x_2 = \psi(x_1)$ when Ω is sufficiently small.

Proposition 4.1. *For every $\varepsilon > 0$, when the support of μ is sufficiently small then*

$$\left(\int_S |\widehat{f}|^2 d\mu^{1-\rho_1} \right)^{1/2} \leq C_{p,\varepsilon} \|f\|_{L^p(\mathbb{R}^3)}, \quad f \in \mathcal{S}(\mathbb{R}^3)$$

whenever $p' \geq 2d + 2$. In particular, this estimate is valid for $p' \geq p'_c$.

The proof of this result will by and large follow the proof of Corollary 1.6 in [14]. By $\{\delta_r\}_{r>0}$ we shall again denote the dilations associated to the principal weight κ . Fixing a suitable smooth cut-off function $\chi \geq 0$ on \mathbb{R}^2 supported in an annulus $\mathcal{A} \subset \mathbb{R}^2$ such that the functions $\chi_k := \chi \circ \delta_{2^k}$ form a partition of unity, we then decompose the measure $\mu^{1-\rho_1}$ dyadically as

$$(4.2) \quad \mu^{1-\rho_1} = \sum_{k \geq k_0} \mu_k,$$

where $\mu_k := \mu^{\chi_k(1-\rho_1)}$. Let us extend the dilations δ_r to \mathbb{R}^3 by putting

$$\delta_r^e(x_1, x_2, x_3) := (r^{\kappa_1} x_1, r^{\kappa_2} x_2, r x_3).$$

We re-scale the measure μ_k by defining $\mu_{0,(k)} := 2^{-k} \mu_k \circ \delta_{2^{-k}}^e$, i.e.,

$$(4.3) \quad \langle \mu_{0,(k)}, f \rangle = 2^{|\kappa|k} \langle \mu_k, f \circ \delta_{2^k}^e \rangle = \int_{(\mathbb{R}_+)^2} f(x, \phi^k(x)) \eta(\delta_{2^{-k}} x) \chi(x) (1 - \rho_1(x_1, x_2)) dx,$$

with

$$(4.4) \quad \phi^k(x) := 2^k \phi(\delta_{2^{-k}} x) = \phi_\kappa(x) + \text{error terms of order } O(2^{-\delta k}),$$

where $\delta > 0$. Recall here that the principal part ϕ_{pr} of ϕ agrees with ϕ_κ . This shows that the measures $\mu_{0,(k)}$ are supported on the smooth hypersurfaces S^k defined as the graph of ϕ^k , their total variations are uniformly bounded, i.e., $\sup_k \|\mu_{0,(k)}\|_1 < \infty$, and that they are approaching the surface carried measure $\mu_{0,(\infty)}$ on S defined by

$$\langle \mu_{0,(\infty)}, f \rangle := \int_{(\mathbb{R}_+)^2} f(x, \phi_\kappa(x)) \eta(0) \chi(x) (1 - \rho_1(x_1, x_2)) dx$$

as $k \rightarrow \infty$. The proof of Corollary 1.6 in [14], which is based on a classical result by A. Greenleaf [10] which relates uniform estimates for the Fourier transform of a surface carried measure to L^p - L^2 - Fourier restriction estimates for this measure, as well as on Littlewood-Paley theory, then shows that it is sufficient to verify the following estimate in order to prove Proposition 4.1:

Lemma 4.2. *If $k_0 \in \mathbb{N}$ is sufficiently large, then there exists a constant $C > 0$ such that*

$$|\widehat{\mu_{0,(k)}}(\xi)| \leq C(1 + |\xi|)^{-1/d} \quad \text{for every } \xi \in \mathbb{R}^3, k \geq k_0.$$

We turn to the proof of Lemma 4.2. Assume first that $h_{\text{lin}} = h_{\text{lin}}(\phi) \geq 2$. Then $h(\phi) > 2$ by Assumption 1.2. Thus, in this case, the proof of Lemma 2.3 in [14] shows that indeed the estimate in Lemma 4.2 holds true.

We may therefore assume that $h_{\text{lin}} < 2$, so that ϕ can be assumed to be given by one of the normal forms appearing in Proposition 3.1. Moreover, then $h_{\text{lin}} = d$ is the Newton distance. Let us re-write

$$\widehat{\mu_{0,(k)}}(\xi) = \int_{(\mathbb{R}_+)^2} e^{-i(\xi_1 x_1 + \xi_2 x_2 + \xi_3 \phi^k(x_1, x_2))} \eta(\delta_{2^{-k}} x) \chi(x) (1 - \rho_1(x_1, x_2)) dx,$$

and observe that, by a partition of unity argument, it will suffice to prove the following:

Given any point $v \in \mathcal{A}$ such that

$$(4.5) \quad v_2 - cv_1^m \neq 0,$$

there is neighborhood V of v such that for every bump function $\chi_v \in C^\infty(\mathbb{R}^2)$ supported in V we have

$$(4.6) \quad |J^{\chi_v}(\xi)| \leq C(1 + |\xi|)^{-1/d} \quad \text{for every } \xi \in \mathbb{R}^3, k \geq k_0,$$

where

$$J^{\chi_v}(\xi) := \int_{(\mathbb{R}_+)^2} e^{-i(\xi_1 x_1 + \xi_2 x_2 + \xi_3 \phi^k(x_1, x_2))} \eta(\delta_{2^{-k}} x) \chi_v(x) dx.$$

To prove this, we shall distinguish the cases a and b from Proposition 3.1.

Case a (ϕ of type A). In this case, we see that $\kappa = (\frac{1}{2m}, \frac{1}{2})$ and

$$\phi_\kappa(x_1, x_2) = \phi_{\text{pr}}(x_1, x_2) = b(0, 0)(x_2 - cx_1^m)^2,$$

so that $\frac{1}{d} = \frac{1}{2} + \frac{1}{2m}$. After applying a suitable linear change of coordinates (and possibly complex conjugation to $J^{\chi_v}(\xi)$), we may assume that $b(0, 0) = 1$. Then, the Hessian of ϕ_κ is given by

$$\text{Hess}(\phi_\kappa)(x_1, x_2) := -4m(m-1)cx_1^{m-2}(x_2 - cx_1^m).$$

Therefore, by (4.5), if $m = 2$, or $v_1 \neq 0$, then $\text{Hess}(\phi_\kappa)(v) \neq 0$. In this case, in view of (4.4) we can apply the method of stationary phase for phase functions depending on small parameters and easily obtain

$$|J^{\chi_v}(\xi)| \leq C(1 + |\xi|)^{-1} \quad \text{for every } \xi \in \mathbb{R}^3, k \geq k_0,$$

provided V is sufficiently small and k_0 sufficiently large. Since $d \geq 1$, this yields (4.6).

We are left with the case where $m > 2$ and $v_1 = 0$. Since $v = (v_1, v_2) \in \mathcal{A}$, this implies that $v_2 \neq 0$.

Putting $\tilde{\phi}^k(y_1, y_2) := \phi^k(y_1, v_2 + y_2)$, we may re-write $J^{\chi_v}(\xi)$ as

$$J^{\chi_v}(\xi) = e^{-iv_2\xi_2} \int_{(\mathbb{R}_+)^2} e^{-i(\xi_1 y_1 + \xi_2 y_2 + \xi_3 \tilde{\phi}^k(y_1, y_2))} \eta(\delta_{2^{-k}}(y_1, v_2 + y_2)) \tilde{\chi}_0(y) dy,$$

where $\tilde{\chi}_0$ is now supported in a sufficiently small neighborhood of the origin. But,

$$\begin{aligned} \tilde{\phi}^k(y_1, y_2) &= (v_2 + y_2 - cy_1^m)^2 + O(2^{-\delta k}) \\ &= v_2^2 + 2v_2 y_2 + \left(y_2^2 - 2cv_2 y_1^m + c^2 y_1^{2m} - 2cy_2 y_1^m\right) + O(2^{-\delta k}). \end{aligned}$$

The main term here is $(y_2^2 - 2cv_2 y_1^m)$, which shows that the phase has a singularity of type A_{m-1} .

By means of a linear change of variables in ξ -space, which replaces $\xi_2 + 2v_2\xi_3$ by ξ_2 , we may thus reduce to assuming that the complete phase in the oscillatory integral $J^{\chi_v}(\xi)$ is given by

$$\xi_1 y_1 + \xi_2 y_2 + \xi_3 \left(y_2^2 - 2cv_2 y_1^m + c^2 y_1^{2m} - 2cy_2 y_1^m + O(2^{-\delta k}) \right).$$

We claim that

$$|J^{\chi_v}(\xi)| \leq C(1 + |\xi|)^{-(\frac{1}{2} + \frac{1}{m})} \quad \text{for every } \xi \in \mathbb{R}^3, k \geq k_0,$$

which is better than (4.6).

Indeed, if

$$|\xi_3| \ll \max\{|\xi_1|, |\xi_2|\},$$

then this follows easily by integration by parts, so let us assume that

$$|\xi_3| \geq M \max\{|\xi_1|, |\xi_2|\}$$

for some constant $M > 0$. Then $|\xi_3| \sim |\xi|$. Consequently, by applying first the method of stationary phase to the integration in y_2 , and then van der Corput's estimate to the y_1 integration, we obtain the estimate above. Observe here that these types of estimates are stable under small, smooth perturbations.

Case b (ϕ of type D). In this case, we see that $\kappa = (\frac{1}{2m+1}, \frac{m}{2m+1})$ and

$$\phi_\kappa(x_1, x_2) = \phi_{\text{pr}}(x_1, x_2) = g(0, 0)x_1(x_2 - cx_1^m)^2,$$

so that $\frac{1}{d} = \frac{m+1}{2m+1}$. Again, we may assume without loss of generality that $g(0, 0) = 1$, so that

$$\phi_\kappa(x_1, x_2) = x_1 x_2^2 - 2cx_1^{m+1}x_2 + c^2 x_1^{2m+1}.$$

Straight-forward computations show that

$$\begin{aligned} \partial_1^2 \phi_\kappa(x) &= -2cm(m+1)x_1^{m-1}x_2 + c^2 2m(2m+1)x_1^{2m-1}, \\ \partial_1 \partial_2 \phi_\kappa(x) &= 2x_2 - 2c(m+1)x_1^m, \quad \partial_2^2 \phi_\kappa(x) = 2x_1, \end{aligned}$$

hence

$$\text{Hess}(\phi_\kappa)(v) := -4(x_2 - cx_1^m) \left(x_2 + c(m^2 - m - 1)x_1^m \right).$$

In view of (4.5), we see that $\text{Hess}(\phi_\kappa)(v) \neq 0$, if $v_2 + c(m^2 - m - 1)v_1^m \neq 0$, so that we can again estimate $J^{\chi_v}(\xi)$ by means of the method of stationary phase.

Let us therefore assume that $\text{Hess}(\phi_\kappa)(v) = 0$, i.e.,

$$(4.7) \quad v_2 = -c(m^2 - m - 1)v_1^m.$$

Observe that then $v_1 \neq 0, v_2 \neq 0$. Denote by

$$P_j(y) := \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial^\alpha \phi_\kappa(v) y^\alpha$$

the homogeneous Taylor polynomial of ϕ_κ of degree j , centered at v . Then clearly

$$P_2(y) = v_1 \left(y_2 + (v_2 - c(m+1)v_1^m)y_1/v_1 \right)^2 = v_1 \left(y_2 - cm^2 v_1^{m-1} y_1 \right)^2.$$

Moreover, by (4.7)

$$\begin{aligned} P_3(y) &= -y_1 \left(\frac{1}{3} c^2 m^2 (m^3 - m^2 + 2m + 1) v_1^{2m-2} y_1^2 - cm(m+1) v_1^{m-1} y_1 y_2 + y_2^2 \right) \\ &= -y_1 Q(y). \end{aligned}$$

Passing to the linear coordinates $z_1 := y_1$, $z_2 := y_2 - cm^2 v_1^{m-1} y_1$, one finds that

$$P_2 = v_1 z_2^2, \quad P_3 = -z_1 \tilde{Q}(z),$$

where again $\tilde{Q} = z_2^2 + 2\beta_1 z_1 z_2 + \beta_2 z_1^2$ is again a quadratic form. Moreover, straightforward computations show that

$$\beta_2 = \frac{c^2}{3} m^2 (m-1) (m^2 - 1) v_1^{2m-2} \neq 0.$$

Applying Taylor's formula, we thus find that, in the coordinates z ,

$$\tilde{\phi}(z) := \phi_\kappa(v_1 + y_1, v_2 + y_2) = c_0 + c_1 z_1 + c_2 z_2 + (v_1 z_2^2 - \beta_2 z_1^3) - (z_1 z_2^2 + 2\beta_1 z_1^2 z_2) + O(|z|^4).$$

Let us put $\phi^v(z) := \phi(z) - (c_0 + c_1 z_1 + c_2 z_2)$, so that $\phi^v(0, 0) = 0$, $\nabla \phi^v(0, 0) = 0$. Then one finds that the principal part of ϕ^v is given by

$$\phi_{\text{pr}}^v(z) = v_1 z_2^2 - \beta_2 z_1^3, \quad \text{where } \beta_2 \neq 0.$$

We can now argue in a very similar way as in the previous case. Indeed, by passing for the variables x in the integral defining $J^{xv}(\xi)$, and then applying first the method of stationary phase to the integration in z_2 , and subsequently van der Corput's estimate to the z_1 integration (in the case where $|\xi_3| \geq M \max\{|\xi_1|, |\xi_2|\}$), we obtain the estimate

$$|J^{xv}(\xi)| \leq C(1 + |\xi|)^{-(\frac{1}{2} + \frac{1}{3})} \quad \text{for every } \xi \in \mathbb{R}^3, k \geq k_0.$$

Again, this is a stronger estimate than (4.6), since here

$$\frac{1}{d} = \frac{1}{2} + \frac{1}{4m+2} \leq \frac{1}{2} + \frac{1}{3}.$$

The proof of Proposition 4.1 is thus complete.

We are thus left with proving Fourier restriction estimates for the measure μ^{ρ_1} which is supported in the small neighborhood (4.1) of the principal root jet. Our main goal will thus be to prove the following

Proposition 4.3. *Assume that ϕ satisfies the assumptions of Theorem 1.7. If $\varepsilon > 0$ is sufficiently small, then we have*

$$\left(\int_S |\widehat{f}|^2 d\mu^{\rho_1} \right)^{1/2} \leq C_{p,\varepsilon} \|f\|_{L^p(\mathbb{R}^3)}, \quad f \in \mathcal{S}(\mathbb{R}^3)$$

whenever $p' \geq p'_c$.

In combination with Proposition 4.1 this will conclude the proof of Theorem 1.7. Notice that by interpolation with the trivial L^1 - L^2 -restriction estimate, it will suffice to prove this for $p = p_c$.

We shall distinguish between the cases where $h_{\text{lin}} < 2$, and where $h_{\text{lin}} \geq 2$, since their treatments will require somewhat different approaches. Moreover, when $h_{\text{lin}} \geq 5$, some arguments simplify substantially compared to the case where $2 < h_{\text{lin}} < 5$, since we can then apply restriction estimates for curves with non-vanishing curvature originating from seminal work by S.W. Drury, so that we shall also distinguish between those sub-cases.

5. THE CASE WHEN $h_{\text{lin}}(\phi) < 2$

In this case, we may assume that ϕ is given by one of the normal forms in Proposition 3.1. Recall from Corollary 3.3 that then $p'_c = 2d + 2$. Recall also that, because we are assuming of Condition (R), the term b_0 in (3.1) respectively (3.2) vanishes identically if ϕ is of type A_∞ or D_∞ (cf. Remark 3.2 (c)).

In a first step, we shall follow the arguments from the preceding section and decompose the measure μ^{ρ_1} dyadically by means of the dilations associated to the principal weight κ . Applying subsequent re-scalings, we may then reduce ourselves by means of Littlewood-Paley theory to proving the following uniform restriction estimates (5.3):

For $k \in \mathbb{N}$ denote by ν_k the measure given by

$$(5.1) \quad \langle \nu_k, f \rangle = 2^{|\kappa|k} \langle \mu_k, f \circ \delta_{2^{-k}}^e \rangle = \int_{(\mathbb{R}_+)^2} f(x, \phi^k(x)) \eta(\delta_{2^{-k}} x) \chi(x) \rho_1(x_1, x_2) dx,$$

where ϕ^k is again given by (4.4). Observe that

$$(5.2) \quad x_1 \sim 1 \sim x_2$$

in the support of the integrand. Recall also from (1.9) that

$$\phi(x_1, x_2) = \phi^a(x_1, x_2 - \psi(x_1)),$$

where according to (1.7) we may write

$$\psi(x_1) = x_1^m \omega(x_1), \quad (m \geq 2),$$

with a smooth function ω satisfying $\omega(0) \neq 0$.

Then, if $\varepsilon > 0$ and δ are chosen sufficiently small, there are constants $C_\varepsilon > 0$ and $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$

$$(5.3) \quad \left(\int |\widehat{f}|^2 d\nu_k \right)^{1/2} \leq C_\varepsilon \|f\|_{L^{p_c}(\mathbb{R}^3)}, \quad f \in \mathcal{S}(\mathbb{R}^3).$$

In order to prove this estimate, observe that ϕ^k can be written in the form

$$(5.4) \quad \phi(x, \delta) := \tilde{b}(x_1, x_2, \delta_1, \delta_2) \left(x_2 - x_1^m \omega(\delta_1 x_1) \right)^2 + \delta_0 x_1^n \beta(\delta_1 x_1),$$

where

$$\delta = (\delta_0, \delta_1, \delta_2) = (2^{-(n\kappa_1-1)k}, 2^{-\kappa_1 k}, 2^{-\kappa_2 k})$$

are small parameters which tend to 0 as k tends to infinity, and where \tilde{b} is a smooth function in all variables given by

$$(5.5) \quad \tilde{b}(x_1, x_2, \delta_1, \delta_2) := \begin{cases} b(\delta_1 x_1, \delta_2 x_2), & \text{for } \phi \text{ of type } A, \\ x_1 b_1(\delta_1 x_1, \delta_2 x_2) + \delta_1^{2m-1} x_2^2 b_2(\delta_2 x_2), & \text{for } \phi \text{ is type } D. \end{cases}$$

Recall that $\delta_0 = 0$ when ϕ is of type A_∞ or D_∞ . Recall also that here $x_1 \sim 1 \sim x_2$, and notice that

$$\omega(0) \neq 0, \text{ and } \tilde{b}(x_1, x_2, 0, 0) \sim 1.$$

It is thus easily seen by means of a partition of unity argument that it will suffice to prove the following proposition in order to verify (5.3).

Proposition 5.1. *Let $\phi(x, \delta)$ be as in (5.4). Then, for every point $v = (v_1, v_2)$ such that $v_1 \sim 1$ and $v_2 = v_1^m \omega(0)$, there exists a neighborhood V of v in $(\mathbb{R}_+)^2$ such that for every cut-off function $\eta \in \mathcal{D}(V)$, the measure ν_δ given by*

$$\langle \nu_\delta, f \rangle := \int f(x, \phi(x, \delta)) \eta(x_1, x_2) dx$$

satisfies a restriction estimate

$$(5.6) \quad \left(\int |\widehat{f}|^2 d\nu_\delta \right)^{1/2} \leq C_\eta \|f\|_{L^{pc}(\mathbb{R}^3)}, \quad f \in \mathcal{S}(\mathbb{R}^3),$$

provided δ is sufficiently small, with a constant C_η which depends only on some C^k -norm of η .

In order to prove this proposition, we shall perform yet another dyadic decomposition, this time with respect to the x_3 -variable. A straight-forward modification of the proof of Corollary 1.6 in [14] then allows to reduce the proof again by means of Littlewood-Paley theory to uniform restriction estimates for the following family of measures:

$$(5.7) \quad \langle \nu_{\delta,j}, f \rangle := \int f(x, \phi(x, \delta)) \chi(2^{2j} \phi(x, \delta)) \eta(x_1, x_2) dx.$$

Here, $\chi \in \mathcal{D}(\mathbb{R})$ is any fixed, non-negative smooth bump-function supported in $(-2, -1/2) \cup (1/2, 2)$ such that $\chi \equiv 1$ in a neighborhood of the points -1 and 1 . Notice that $\nu_{\delta,j}$ is supported where $|\phi(x, \delta)| \sim 2^{-2j}$. I.e., in place of (5.6), it will be sufficient to prove an analogous uniform estimate

$$(5.8) \quad \left(\int |\widehat{f}|^2 d\nu_{\delta,j} \right)^{1/2} \leq C_\eta \|f\|_{L^{pc}(\mathbb{R}^3)}, \quad f \in \mathcal{S}(\mathbb{R}^3),$$

for all $j \in \mathbb{N}$ sufficiently big, say $j \geq j_0$, where the constant C_η does neither depend on δ , nor on j .

In order to verify (5.8), we shall distinguish three cases, depending on the size of $2^{2j} \delta_0$.

5.1. **The sub-case where $2^{2j}\delta_0 \gg 1$.** Observe first that if j is sufficiently large, then by (5.4) and since $x_1 \sim 1$, $\nu_{\delta,j} = 0$ unless $\tilde{b}(v, \delta_1, \delta_2)$ and $\beta(0)$ have opposite signs. So, let us for instance assume that $\tilde{b}(x_1, x_2, \delta_1, \delta_2) > 0$ and $\beta(\delta_1 x_1) < 0$ on the support of η . Then $\tilde{\beta} := -\beta > 0$, and we may re-write

$$2^{2j}\phi(x, \delta) = 2^{2j}\tilde{b}(x_1, x_2, \delta_1, \delta_2) \left(x_2 - x_1^m \omega(\delta_1 x_1) \right)^2 - 2^{2j}\delta_0 x_1^n \tilde{\beta}(\delta_1 x_1).$$

We introduce new coordinates y by putting $y_1 := x_1$ and $y_2 := 2^{2j}\phi(x, \delta)$. Solving for x_2 , one easily finds that

$$(5.9) \quad x_2 = \tilde{b}_1 \left(y_1, \sqrt{2^{-2j}y_2 + \delta_0 y_1^n \tilde{\beta}(\delta_1 y_1)}, \delta_1, \delta_2 \right) \sqrt{2^{-2j}y_2 + \delta_0 y_1^n \tilde{\beta}(\delta_1 y_1)} + y_1^m \omega(\delta_1 y_1),$$

where \tilde{b}_1 has similar properties like \tilde{b} . Moreover, by the support properties of the amplitude $\chi(2^{2j}\phi(x, \delta))\eta(x_1, x_2)$, we see that also for the new coordinates we have $y_1 \sim 1 \sim y_2$, and that we can re-write

$$\langle \nu_{\delta,j}, f \rangle = \frac{2^{-2j}}{\sqrt{\delta_0}} \int f \left(y_1, \phi(y, \delta, j), 2^{-2j}y_2 \right) a(y, \delta, j) \chi(y_1) \chi(y_2) dy,$$

with a cut-off function χ as before, and where $a(y, \delta, j)$ is smooth in y and δ , with C^k -norms uniformly bounded in δ and j , and where

$$(5.10) \quad \phi(x, \delta, j) := \tilde{b}_1 \left(x_1, \sqrt{2^{-2j}x_2 + \delta_0 x_1^n \tilde{\beta}(\delta_1 x_1)}, \delta_1, \delta_2 \right) \sqrt{2^{-2j}x_2 + \delta_0 x_1^n \tilde{\beta}(\delta_1 x_1)} + x_1^m \omega(\delta_1 x_1).$$

We have re-named the variable y to x here, since if we define the measure $\tilde{\nu}_{\delta,j}$ by

$$(5.11) \quad \langle \tilde{\nu}_{\delta,j}, f \rangle := \int f \left(x_1, \phi(x, \delta, j), x_2 \right) a(x, \delta, j) \chi(x_1) \chi(x_2) dx,$$

then the restriction estimate (5.8) for the measure $\nu_{\delta,j}$ is equivalent to the following restriction estimate for the measure $\tilde{\nu}_{\delta,j}$:

$$(5.12) \quad \int |\widehat{f}|^2 d\tilde{\nu}_{\delta,j} \leq C_\eta \sqrt{\delta_0} 2^{2j(1-\frac{2}{pc'})} \|f\|_{L^{pc}(\mathbb{R}^3)}^2, \quad f \in \mathcal{S}(\mathbb{R}^3)$$

for all $j \in \mathbb{N}$ sufficiently big, say $j \geq j_0$, where the constant C_η does neither depend on δ , nor on j .

Formula (5.11) shows that the Fourier transform of the measure $\tilde{\nu}_{\delta,j}$ can be expressed as an oscillatory integral

$$(5.13) \quad \widehat{\tilde{\nu}_{\delta,j}}(\xi) = \int e^{-i\Phi(x, \delta, j, \xi)} a(x, \delta, j) \chi(x_1) \chi(x_2) dx,$$

where the complete phase function Φ is given by

$$(5.14) \quad \Phi(x, \delta, j, \xi) := \xi_2 \phi(x, \delta, j) + \xi_3 x_2 + \xi_1 x_1.$$

Finally, we shall perform a Littlewood- Paley decomposition of the measure $\tilde{\nu}_{\delta,j}$ in each coordinate. To this end, we fix again a suitable smooth cut-off function $\chi_1 \geq 0$

on \mathbb{R} supported in $(-2, -1/2) \cup (1/2, 2)$ such that the functions $\chi_k(t) := \chi_1(2^{1-k}t)$, $k \in \mathbb{N} \setminus \{0\}$, in combination with a suitable smooth function χ_0 supported in $(-1, 1)$, form a partition of unity, i.e.,

$$(5.15) \quad \sum_{k=0}^{\infty} \chi_k(t) = 1 \quad \text{for all } t \in \mathbb{R}.$$

For every multi-index $k = (k_1, k_2, k_3) \in \mathbb{N}^3$, we put

$$(5.16) \quad \chi_k(\xi) := \chi_{k_1}(\xi_1) \chi_{k_2}(\xi_2) \chi_{k_3}(\xi_3),$$

and finally define the smooth functions $\nu_{k,j}$ by

$$\widehat{\nu_{k,j}}(\xi) := \chi_k(\xi) \widehat{\tilde{\nu}_{\delta,j}}(\xi).$$

In order to defray the notation, we have suppressed here the dependency of this smooth function on the small parameters δ . We then find that

$$(5.17) \quad \tilde{\nu}_{\delta,j} = \sum_{k \in \mathbb{N}^3} \nu_{k,j},$$

in the sense of distributions. To simplify the subsequent discussion, we shall concentrate on those measures $\nu_{k,j}$ for which none of its components k_i 's are zero, since the remaining cases where for instance k_i is zero can be dealt with in the same way as the corresponding cases where $k_i \geq 1$ is small.

Now, if $1 \leq \lambda_i = 2^{k_i-1}$, $i = 1, 2, 3$, are dyadic numbers, we shall accordingly write ν_j^λ in place of $\nu_{k,j}$, i.e.,

$$(5.18) \quad \widehat{\nu_j^\lambda}(\xi) = \chi_1\left(\frac{\xi_1}{\lambda_1}\right) \chi_1\left(\frac{\xi_2}{\lambda_2}\right) \chi_1\left(\frac{\xi_3}{\lambda_3}\right) \widehat{\tilde{\nu}_{\delta,j}}(\xi).$$

Note that

$$(5.19) \quad |\xi_i| \sim \lambda_i, \quad \text{on } \text{supp } \widehat{\nu_j^\lambda}.$$

Moreover, by (5.11),

$$(5.20) \quad \begin{aligned} \nu_j^\lambda(x) = \lambda_1 \lambda_2 \lambda_3 \int \check{\chi}_1\left(\lambda_1(x_1 - y_1)\right) \check{\chi}_1\left(\lambda_2(x_2 - \phi(y, \delta, j))\right) \\ \check{\chi}_1\left(\lambda_3(x_2 - y_2)\right) a(y, \delta, j) \chi(y_1) \chi(y_2) dy, \end{aligned}$$

where \check{f} denotes the inverse Fourier transform of f .

We begin by estimating the Fourier transform of ν_j^λ . To this end, we first integrate in x_1 in (5.11), and then in x_2 , assuming that (5.19) holds true. We shall concentrate on those ν_j^λ for which

$$(5.21) \quad \lambda_1 \sim \lambda_2 \sim \sqrt{\delta_0} 2^{2j} \lambda_3.$$

In all other cases, the phase has no critical point on the support of the amplitude, and

we obtain much faster Fourier decay estimates by repeated integrations by parts, so that the corresponding terms can be considered as error terms. Observe also that

$$\frac{\partial^2}{\partial x_2^2} \Phi(x, \delta, j, \xi) \sim \lambda_2 \delta_0^{-3/2} 2^{-4j}$$

on the support of the amplitude. We therefore distinguish two sub-cases.

1. Case: $1 \leq \lambda_1 \lesssim \delta_0^{3/2} 2^{4j}$. In this case we cannot gain from the integration in x_2 but, by applying van der Corput's lemma (or the method of stationary phase) in x_1 we obtain

$$(5.22) \quad \|\widehat{\nu_j^\lambda}\|_\infty \lesssim \frac{1}{\lambda_1^{1/2}}.$$

2. Case: $\lambda_1 \gg \delta_0^{3/2} 2^{4j}$. Then, by first applying the method of stationary phase to the integration in x_1 , and subsequently applying the classical van der Corput lemma (or Lemma 2.2, with $M = 2$) to the integration in x_2 , we obtain

$$(5.23) \quad \|\widehat{\nu_j^\lambda}\|_\infty \lesssim \frac{1}{\lambda_1^{1/2}} \frac{1}{(\lambda_2 \delta_0^{-3/2} 2^{-4j})^{1/2}} \lesssim \frac{\delta_0^{3/4} 2^{2j}}{\lambda_1}.$$

Next, from (5.20), we trivially obtain the following estimate for the L^∞ -norm of ν_j^λ :

$$(5.24) \quad \|\nu_j^\lambda\|_\infty \lesssim \lambda_2 \sim \lambda_1,$$

in Case 1 as well as in Case 2. All these estimates are uniform in δ , for δ sufficiently small.

For each of the measures ν_j^λ , we can now obtain suitable restriction estimates by applying the usual approach. Let us denote by $T_{\delta,j}$ the convolution operator

$$T_{\delta,j} : \varphi \mapsto \varphi * \widehat{\nu_{\delta,j}},$$

and similarly by T_j^λ the convolution operator

$$T_j^\lambda : \varphi \mapsto \varphi * \widehat{\nu_j^\lambda}.$$

Formally, by (5.17), $T_{\delta,j}$ decomposes as

$$(5.25) \quad T_{\delta,j} = \sum_{k \in \mathbb{N}^3} T_j^{2^k},$$

if 2^k represents the vector $2^k := (2^{k_1}, 2^{k_2}, 2^{k_3})$ (with a suitably modified definition of $T_j^{2^k}$ when one of the components k_i is zero). If we denote by $\|T\|_{p \rightarrow q}$ the norm of T as an operator from L^p to L^q , then clearly $\|T_j^\lambda\|_{1 \rightarrow \infty} = \|\widehat{\nu_j^\lambda}\|_\infty$ and $\|T_j^\lambda\|_{2 \rightarrow 2} = \|\nu_j^\lambda\|_\infty$.

The estimates (5.22) - (5.24) thus yield the following bounds:

$$\|T_j^\lambda\|_{1 \rightarrow \infty} \lesssim \begin{cases} \lambda_1^{-1/2}, & \text{if } 1 \leq \lambda_1 \lesssim \delta_0^{3/2} 2^{4j}, \\ \frac{\delta_0^{3/4} 2^{2j}}{\lambda_1}, & \text{if } \lambda_1 \gg \delta_0^{3/2} 2^{4j}, \end{cases}$$

and $\|T_j^\lambda\|_{2 \rightarrow 2} \lesssim \lambda_1$. Interpolating these estimates, where $1/p'_c = (1-\theta)/\infty + \theta/2 = \theta/2$, we find that

$$(5.26) \quad \|T_j^\lambda\|_{p_c \rightarrow p'_c} \lesssim \begin{cases} \lambda_1^{\frac{3\theta-1}{2}}, & \text{if } 1 \leq \lambda_1 \lesssim \delta_0^{3/2} 2^{4j}, \\ \delta_0^{\frac{3}{4}(1-\theta)} 2^{2(1-\theta)j} \lambda_1^{2\theta-1}, & \text{if } \lambda_1 \gg \delta_0^{3/2} 2^{4j}, \end{cases}$$

where

$$(5.27) \quad \theta := \begin{cases} \frac{m+1}{3m+1}, & \text{if } \phi \text{ is of type } A, \\ \frac{m+1}{3m+2}, & \text{if } \phi \text{ is of type } D. \end{cases}$$

The following lemma will be useful in the sequel.

Lemma 5.2. (a) *If ϕ is of type A, then*

$$2\theta - 1 = -\frac{m-1}{3m+1}, \quad 3\theta - 1 = \frac{2}{3m+1}, \quad 3\theta - 2 = -\frac{2m-1}{3m+1}, \quad 5\theta - 2 = \frac{3-m}{6m+2}, \quad 11\theta - 5 = -2\frac{2m-3}{3m+1}$$

and $14\theta - 5 = \frac{9-m}{3m+1}$. Moreover, $1 - \frac{4}{p'_c} = \frac{m-1}{3m+1}$.

(b) *If ϕ is of type D, then*

$$2\theta - 1 = -\frac{m}{3m+2}, \quad 3\theta - 1 = \frac{1}{3m+2}, \quad 3\theta - 2 = -\frac{3m+1}{3m+2}, \quad 5\theta - 2 = -\frac{m-1}{6m+4}, \quad 11\theta - 5 = \frac{1-4m}{3m+2}$$

and $14\theta - 5 = \frac{4-m}{3m+2}$. Moreover, $1 - \frac{4}{p'_c} = \frac{m}{3m+2}$.

Now, the main contributions to the series (5.25) come from those dyadic $\lambda = 2^k$ for which $\lambda_1 \sim \lambda_2 \sim \sqrt{\delta_0} 2^{2j} \lambda_3$. Under these relations, for λ_1 given, λ_2 and λ_3 may only vary in a finite set whose cardinality is bounded by a fixed number. This shows that, up to an easily bounded error term,

$$\|T_{\delta,j}\|_{p_c \rightarrow p'_c} \lesssim \sum_{\lambda_1=2}^{\delta_0^{3/2} 2^{4j}} \lambda_1^{\frac{3\theta-1}{2}} + \sum_{\lambda_1 > \delta_0^{3/2} 2^{4j}} \delta_0^{\frac{3}{4}(1-\theta)} 2^{2(1-\theta)j} \lambda_1^{(2\theta-1)}.$$

Here, and in the sequel, summation over λ_1, λ_2 etc. means that we sum over dyadic numbers λ_1, λ_2 etc. only. Now, by Lemma 5.2,

$$(5.28) \quad 2\theta - 1 < 0 \text{ and } 0 \leq 3\theta - 1 \leq 1,$$

which yields

$$\|T_{\delta,j}\|_{p_c \rightarrow p'_c} \lesssim \delta_0^{\frac{3}{4}(3\theta-1)} 2^{(3\theta-1)2j}.$$

Applying the usual T^*T -argument, we thus need to prove that

$$\delta_0^{\frac{3}{4}(3\theta-1)} 2^{(3\theta-1)2j} \leq C \sqrt{\delta_0} 2^{2j(1-\frac{2}{p'_c})}$$

in order to verify that the restriction estimate (5.12) holds true for $p = p_c = 2d + 2$. However, since $2/p'_c = \theta$, the previous estimate is equivalent to

$$2^{2j(4\theta-2)} \leq C \delta_0^{\frac{5-9\theta}{4}}.$$

But, since $2^{2j}\delta_0 \gg 1$ and $2\theta - 1 < 0$, we see that $2^{2j(4\theta-2)} \leq C\delta_0^{2-4\theta}$, and therefore we only have to verify that $2 - 4\theta \geq (5 - 9\theta)/4$, i.e.,

$$7\theta \leq 3.$$

This is obvious by (5.27), and we thus have verified the restriction estimate (5.8) in this sub-case.

There remains the case $2^{2j}\delta_0 \leq C$, where C is a fixed, possibly large constant.

Observe that the change of variables $(x_1, x_2) \mapsto (x_1, x_2 + x_1^m \omega(\delta_1 x_1))$ and subsequent scaling in x_2 by the factor 2^{-j} allows to write

$$\langle \nu_{\delta,j}, f \rangle = 2^{-j} \int f\left(x_1, 2^{-j}x_2 + x_1^m \omega(\delta_1 x_1), 2^{-2j}\phi^a(x, \delta, j)\right) a(x, \delta, j) dx,$$

where here

$$\phi^a(x, \delta, j) := \tilde{b}(x_1, 2^{-j}x_2 + x_1^m \omega(\delta_1 x_1), \delta_1, \delta_2)x_2^2 + 2^{2j}\delta_0 x_1^n \beta(\delta_1 x_1),$$

and

$$a(x, \delta, j) := \chi(\phi^a(x, \delta, j)) \eta(x_1, 2^{-j}x_2 + x_1^m \omega(\delta_1 x_1)).$$

Let us here introduce the re-scaled measure $\tilde{\nu}_{\delta,j}$ by

$$(5.29) \quad \langle \tilde{\nu}_{\delta,j}, f \rangle := \int f\left(x_1, 2^{-j}x_2 + x_1^m \omega(\delta_1 x_1), \phi^a(x, \delta, j)\right) a(x, \delta, j) dx.$$

Then, it is easy to see by means of a scaling in the variable x_3 by the factor 2^{-2j} that the restriction estimate (5.8) for the measure $\nu_{\delta,j}$ is equivalent to the following restriction estimate for the measure $\tilde{\nu}_{\delta,j}$:

$$(5.30) \quad \int_S |\widehat{f}|^2 d\tilde{\nu}_{\delta,j} \leq C_\eta 2^{(1-\frac{4}{p_c})j} \|f\|_{L^{p_c}(\mathbb{R}^3)}^2, \quad f \in \mathcal{S}(\mathbb{R}^3),$$

for all $j \in \mathbb{N}$ sufficiently big, say $j \geq j_0$, where the constant C_η does neither depend on δ , nor on j .

In order to prove (5.30), we again distinguish two sub-cases.

5.2. The sub-case where $2^{2j}\delta_0 \ll 1$. Notice that here the phase $\phi^a(x, \delta, j)$ is a small perturbation of $\tilde{b}(v, 0, 0)x_2^2$, where $\tilde{b}(v, 0, 0) \sim 1$. This shows that also in the new coordinates, $x_1 \sim 1 \sim x_2$ on the support of the amplitude a , which in return implies

$$(5.31) \quad \partial_2 \phi^a(x, \delta, j) \sim 1.$$

We can thus write

$$(5.32) \quad \widehat{\tilde{\nu}_{\delta,j}}(\xi) = \int e^{-i\Phi(x, \delta, j, \xi)} a(x, \delta, j) \chi(x_1) \chi(x_2) dx,$$

where the complete phase function Φ is now given by

$$(5.33) \quad \Phi(x, \delta, j, \xi) := \xi_3 \phi^a(x, \delta, j) + 2^{-j} \xi_2 x_2 + \xi_2 x_1^m \omega(\delta_1 x_1) + \xi_1 x_1,$$

and where χ has similar properties as before.

As in the previous sub-case, we perform a Littlewood- Paley decomposition (5.17) of the the measure $\tilde{\nu}_{\delta,j}$ in each coordinate and define the measure ν_j^λ by (5.18). Then here we have

$$(5.34) \quad \begin{aligned} \nu_j^\lambda(x) = \lambda_1 \lambda_2 \lambda_3 \int \check{\chi}_1 \left(\lambda_1(x_1 - y_1) \right) \check{\chi}_1 \left(\lambda_2(x_2 - 2^{-j}y_2 - y_1^m \omega(\delta_1 y_1)) \right) \\ \check{\chi}_1 \left(\lambda_3(x_3 - \phi^a(y, \delta, j)) \right) a(y, \delta, j) \chi(y_1) \chi(y_2) dy, \end{aligned}$$

where \check{f} denotes the inverse Fourier transform of f .

We begin by estimating the Fourier transform of ν_j^λ . To this end, we first integrate in x_2 in (5.32), and then in x_1 . We may assume that (5.19) holds true. Since then the phase function Φ has no critical point in x_2 unless $\lambda_3 \sim 2^{-j}\lambda_2$, and similarly in x_1 , unless $\lambda_2 \sim \lambda_1$, we shall concentrate on those ν_j^λ for which

$$(5.35) \quad \lambda_1 \sim \lambda_2 \quad \text{and} \quad 2^{-j}\lambda_2 \sim \lambda_3.$$

In all other cases, we obtain much faster Fourier decay estimates by repeated integrations by parts, so that the corresponding terms can be considered as error terms.

1. Case: $1 \leq \lambda_1 \leq 2^j$. In this case the phase function has essentially no oscillation in the x_2 variable. But, by applying van der Corput's lemma (or the method of stationary phase) in x_1 we obtain in combination with (5.35) the estimate

$$(5.36) \quad \|\widehat{\nu_j^\lambda}\|_\infty \lesssim \frac{1}{\lambda_1^{1/2}}.$$

2. Case: $\lambda_1 > 2^j$. Observe that in this case, our assumptions imply that $\delta_0 2^{2j} \lambda_3 \ll \lambda_3 \ll \lambda_2$, if $j \geq j_0 \gg 1$. Moreover, depending on the signs of the ξ_i , we may have no critical point, or exactly one non-degenerate critical point, with respect to each of the variables x_2 and x_1 . So, integrating by parts, respectively applying the method of stationary phase in the presence of a critical point, first in x_2 and then in x_1 , we obtain

$$(5.37) \quad \|\widehat{\nu_j^\lambda}\|_\infty \lesssim \frac{2^{j/2}}{\lambda_1}.$$

Next, we estimate the L^∞ -norm of ν_j^λ . To this end, notice that (5.31) shows that we may change coordinates in (5.34) by putting $(z_1, z_2) := (y_1, \phi(y_1, y_2, \delta, j))$. Since the Jacobian of this coordinate change is of order 1, we thus obtain that

$$|\nu_j^\lambda(x)| \lesssim \lambda_1 \lambda_2 \lambda_3 \iint \left| \check{\chi}_1 \left(\lambda_1(x_1 - z_1) \right) \check{\chi}_1 \left(\lambda_3(x_3 - z_2) \right) \tilde{a}(z, \delta, j) \right| dz_1 dz_2,$$

hence

$$(5.38) \quad \|\nu_j^\lambda\|_\infty \lesssim \lambda_2 \sim \lambda_1,$$

in Case 1 as well as in Case 2.

For the operators $T_{\delta,j}$ and T_j^λ which appear in this sub-case, the estimates (5.36) - (5.38) thus yield the following bounds:

$$\|T_j^\lambda\|_{1 \rightarrow \infty} \lesssim \begin{cases} \lambda_1^{-1/2}, & \text{if } 1 \leq \lambda_1 \leq 2^j, \\ 2^{j/2} \lambda_1^{-1}, & \text{if } \lambda_1 > 2^j, \end{cases}$$

and $\|T_j^\lambda\|_{2 \rightarrow 2} \lesssim \lambda_1$. Interpolating these estimates, we find that

$$(5.39) \quad \|T_j^\lambda\|_{p_c \rightarrow p'_c} \lesssim \begin{cases} \lambda_1^{\frac{3\theta-1}{2}}, & \text{if } 1 \leq \lambda_1 \leq 2^j, \\ 2^{\frac{1-\theta}{2}j} \lambda_1^{2\theta-1}, & \text{if } \lambda_1 > 2^j, \end{cases}$$

where θ is again given by (5.27).

Now, in view of (5.35), the main contributions to the series (5.25) comes here from those dyadic $\lambda = 2^k$ for which $\lambda_1 \sim \lambda_2$ and $2^{-j} \lambda_2 \sim \lambda_3$. Thus, up to an easily bounded error term,

$$(5.40) \quad \|T_{\delta,j}\|_{p_c \rightarrow p'_c} \lesssim \sum_{\lambda_1=2}^{2^j} \lambda_1^{\frac{3\theta-1}{2}} + \sum_{\lambda_1=2^{j+1}}^{\infty} 2^{\frac{1-\theta}{2}j} \lambda_1^{2\theta-1} \lesssim 2^{\frac{3\theta-1}{2}j},$$

since, by Lemma 5.2, $2\theta - 1 < 0$.

Consequently, if ϕ is of type A , then we see by means of Lemma 5.2 that

$$\|T_{\delta,j}\|_{p_c \rightarrow p'_c} \lesssim 2^{\frac{1}{3m+1}j} \leq 2^{\frac{m-1}{3m+1}j} = 2^{(1-\frac{4}{p'_c})j}.$$

Similarly, if ϕ is of type D , then

$$\|T_{\delta,j}\|_{p_c \rightarrow p'_c} \lesssim 2^{\frac{1}{6m+4}j} \leq 2^{\frac{m}{6m+2}j} = 2^{(1-\frac{4}{p'_c})j},$$

so that we have verified the restriction estimate (5.30).

This concludes the proof of Proposition 5.1 also in this sub-case.

5.3. The sub-case where $2^{2j}\delta_0 \sim 1$. Notice that here we can no longer conclude that $x_2 \sim 1$ on the support of the amplitude $a(x, \delta, j)$, but only that $|x_2| \lesssim 1$, whereas still $x_1 \sim 1$.

Putting $\sigma := 2^{2j}\delta_0$, and $b^\sharp(x, \delta, j) := \tilde{b}(x_1, 2^{-j}x_2 + x_1^m \omega(\delta_1 x_1), \delta_1, \delta_2)$, we may re-write the complete phase in (5.33) as

$$(5.41) \quad \begin{aligned} \Phi(x, \delta, j, \xi) &= \xi_1 x_1 + \xi_2 x_1^m \omega(\delta_1 x_1) + \xi_3 \sigma x_1^n \beta(\delta_1 x_1) \\ &\quad + 2^{-j} \xi_2 x_2 + \xi_3 b^\sharp(x, \delta, j) x_2^2, \end{aligned}$$

where $\sigma \sim 1$ and $|b^\sharp(x, \delta, j)| \sim 1$, and (5.34) as

$$(5.42) \quad \begin{aligned} \nu_j^\lambda(x) &= \lambda_1 \lambda_2 \lambda_3 \int \check{\chi}_1 \left(\lambda_1 (x_1 - y_1) \right) \check{\chi}_1 \left(\lambda_2 (x_2 - 2^{-j} y_2 - y_1^m \omega(\delta_1 y_1)) \right) \\ &\quad \check{\chi}_1 \left(\lambda_3 (x_3 - b^\sharp(y, \delta, j) y_2^2 - \sigma y_1^n \beta(\delta_1 y_1)) \right) a(y, \delta, j) dy. \end{aligned}$$

Since $\left| \int \check{\chi}_1 \left(\lambda_3(c - t^2) \right) dt \right| \leq C \lambda_3^{-1/2}$, with a constant C which is independent of c , it is easy to see that

$$(5.43) \quad \|\nu_j^\lambda\|_\infty \lesssim \min\{2^j \lambda_3, \lambda_2 \lambda_3^{1/2}\} = \lambda_3^{1/2} \min\{\lambda_2, 2^j \lambda_3^{1/2}\}.$$

Let us again assume (5.19). We shall first integrate in x_1 in order to estimate $\widehat{\nu_j^\lambda}(\xi)$. If one of the quantities λ_1, λ_2 and λ_3 is much bigger than the two other ones, we see that we have no critical point on the support of the amplitude, so that the corresponding terms can again be viewed as error terms. Let us therefore assume that all three are of comparable size, or two of them are of comparable size, and the third one is much smaller. We shall begin with the latter situation, and distinguish various possibilities.

1. Case: $\lambda_1 \sim \lambda_3$ and $\lambda_2 \ll \lambda_1$. In this case, we apply the method of stationary phase to the integration in x_1 , and subsequently van der Corput's estimate to the x_2 -integration and obtain $\|\widehat{\nu_j^\lambda}\|_\infty \lesssim \lambda_1^{-1/2} \lambda_3^{-1/2} \sim \lambda_1^{-1}$.

(a) Assume first that $\lambda_2 \leq 2^j \lambda_1^{1/2}$. Then, by (5.43), $\|\nu_j^\lambda\|_\infty \lesssim \lambda_2 \lambda_1^{1/2}$, and we obtain in a similar way as before by interpolation that

$$\|T_j^\lambda\|_{p_c \rightarrow p'_c} \lesssim \lambda_1^{\frac{3\theta-2}{2}} \lambda_2^\theta.$$

Here, $\frac{3\theta-2}{2} < 0$, because of (5.27). Note next that if $2^j \lambda_1^{1/2} \leq \lambda_1$, i.e., if $\lambda_1 \geq 2^{2j}$, then by our assumptions $\lambda_2 \leq 2^j \lambda_1^{1/2}$, and if $\lambda_1 < 2^{2j}$, then $\lambda_2 \leq \lambda_1$. We thus find that the contributions $T_{\delta,j}^I$ of the operators T_j^λ with λ satisfying the assumptions in (a) to $T_{\delta,j}$ can be estimated by

$$\begin{aligned} \|T_{\delta,j}^I\|_{p_c \rightarrow p'_c} &\lesssim \sum_{\lambda_1=2}^{2^{2j}} \sum_{\lambda_2=2}^{\lambda_1} \lambda_1^{\frac{3\theta-2}{2}} \lambda_2^\theta + \sum_{\lambda_1=2^{2j+1}}^{\infty} \sum_{\lambda_2=2}^{2^j \lambda_1^{1/2}} \lambda_1^{\frac{3\theta-2}{2}} \lambda_2^\theta \\ &\lesssim \sum_{\lambda_1=2}^{2^{2j}} \lambda_1^{\frac{5\theta-2}{2}} + \sum_{\lambda_1=2^{2j+1}}^{\infty} 2^{\theta j} \lambda_1^{2\theta-1}. \end{aligned}$$

But, we have seen that $2\theta - 1 < 0$, so that

$$\|T_{\delta,j}^I\|_{p_c \rightarrow p'_c} \lesssim \max\{j, 2^{(5\theta-2)j}\}.$$

By means of Lemma 5.2, we thus see that if ϕ is of type A and $m < 3$, then $5\theta - 2 > 0$, hence $\|T_{\delta,j}^I\|_{p_c \rightarrow p'_c} \lesssim 2^{\frac{3-m}{3m+1}j} \leq 2^{\frac{m-1}{3m+1}j} = 2^{(1-\frac{4}{p'})j}$. And, if ϕ is of type A and $m \geq 3$, or if ϕ is of type D , then $5\theta - 2 \leq 0$, hence $\|T_{\delta,j}^I\|_{p_c \rightarrow p'_c} \lesssim j \leq 2^{\frac{m-1}{3m+1}j} = 2^{(1-\frac{4}{p_c})j}$, i.e.,

$$(5.44) \quad \|T_{\delta,j}^I\|_{p_c \rightarrow p'_c} \lesssim 2^{(1-\frac{4}{p_c})j}.$$

(b) Assume next that $\lambda_2 > 2^j \lambda_1^{1/2}$. Then, by (5.43), $\|\nu_j^\lambda\|_\infty \lesssim 2^j \lambda_1$, and we obtain in a similar way as before by interpolation that

$$\|T_j^\lambda\|_{p_c \rightarrow p'_c} \lesssim 2^{\theta j} \lambda_1^{2\theta-1}.$$

Observing that we have $\lambda_2 \leq \lambda_1$ and $\lambda_1 > 2^{2j}$ under our assumptions, we see that the contributions $T_{\delta,j}^{II}$ of the operators T_j^λ with λ satisfying the assumptions in (b) to $T_{\delta,j}$ can be estimated by

$$\|T_{\delta,j}^{II}\|_{p_c \rightarrow p'_c} \lesssim \sum_{\lambda_1=2^{2j+1}}^{\infty} \sum_{\lambda_2=2}^{\lambda_1} 2^{\theta j} \lambda_1^{2\theta-1} \lesssim \sum_{\lambda_1=2^{2j+1}}^{\infty} 2^{\theta j} \log \lambda_1 \lambda_1^{2\theta-1} \lesssim j 2^{(5\theta-2)j},$$

so that also

$$(5.45) \quad \|T_{\delta,j}^{II}\|_{p_c \rightarrow p'_c} \lesssim 2^{(1-\frac{4}{p'_c})j}.$$

2. Case: $\lambda_2 \sim \lambda_3$ and $\lambda_1 \ll \lambda_2$. Here, we can estimate $\widehat{\nu_j^\lambda}$ in the same way as in the previous case and obtain $\|\widehat{\nu_j^\lambda}\|_\infty \lesssim \lambda_2^{-1/2} \lambda_3^{-1/2} \sim \lambda_2^{-1}$.

(a) Assume first that $\lambda_2 \leq 2^j \lambda_1^{1/2}$. Then $\lambda_2 \lesssim 2^{2j}$, and we obtain

$$\|T_j^\lambda\|_{p_c \rightarrow p'_c} \lesssim \lambda_2^{\frac{5\theta-2}{2}}.$$

We thus find that the contributions $T_{\delta,j}^{III}$ of the operators T_j^λ with λ satisfying the assumptions in (a) to $T_{\delta,j}$ can be estimated by

$$\|T_{\delta,j}^{III}\|_{p_c \rightarrow p'_c} \lesssim \sum_{\lambda_2=2}^{2^{2j}} \sum_{\lambda_1=2}^{\lambda_2} \lambda_2^{\frac{5\theta-2}{2}} \lesssim \sum_{\lambda_2=2}^{2^{2j}} \log \lambda_2 \lambda_2^{\frac{5\theta-2}{2}}$$

Arguing in a similar way as in case (a) of Case 1, we obtain

$$(5.46) \quad \|T_{\delta,j}^{III}\|_{p_c \rightarrow p'_c} \lesssim 2^{(1-\frac{4}{p'_c})j}.$$

(b) Assume next that $\lambda_2 > 2^j \lambda_1^{1/2}$. Then, by (5.43), $\|\nu_j^\lambda\|_\infty \lesssim 2^j \lambda_1$, and we obtain in a similar way as before by interpolation that

$$\|T_j^\lambda\|_{p_c \rightarrow p'_c} \lesssim 2^{\theta j} \lambda_1^\theta \lambda_2^{\theta-1}.$$

Since here $\lambda_2 \geq \max\{\lambda_1, 2^j \lambda_1^{1/2}\}$, we then find that the contributions $T_{\delta,j}^{IV}$ of the operators T_j^λ with λ satisfying the assumptions in (a) to $T_{\delta,j}$ can be estimated by

$$\|T_{\delta,j}^{IV}\|_{p_c \rightarrow p'_c} \lesssim \sum_{\lambda_1=2}^{2^{2j}} \sum_{\lambda_2 \geq 2^j \lambda_1^{1/2}} 2^{\theta j} \lambda_1^\theta \lambda_2^{\theta-1} + \sum_{\lambda_1=2^{2j}}^{\infty} \sum_{\lambda_2 \geq \lambda_1} 2^{\theta j} \lambda_1^\theta \lambda_2^{\theta-1} \lesssim 2^{(5\theta-2)j}.$$

As before, this implies that

$$(5.47) \quad \|T_{\delta,j}^{IV}\|_{p_c \rightarrow p'_c} \lesssim 2^{(1-\frac{4}{p'_c})j}.$$

3. Case: $\lambda_1 \sim \lambda_2$ and $\lambda_3 \ll \lambda_1$. Notice that the phase Φ has no critical point with respect to x_2 when $2^{-j} \lambda_2 \gg \lambda_3$, so that we shall concentrate on the case only where $\lambda_2 \lesssim 2^j \lambda_3$. Then we can estimate $\widehat{\nu_j^\lambda}$ in the same way as in the previous cases and obtain $\|\widehat{\nu_j^\lambda}\|_\infty \lesssim \lambda_1^{-1/2} \lambda_3^{-1/2}$.

(a) **Assume first that** $\lambda_1 \leq 2^j \lambda_3^{1/2}$. Then $\lambda_3 \lesssim 2^{2j}$, and we obtain

$$\|T_j^\lambda\|_{p_c \rightarrow p'_c} \lesssim \lambda_1^{\frac{3\theta-2}{2}} \lambda_3^{\frac{2\theta-2}{2}}.$$

We thus find that the contributions $T_{\delta,j}^V$ of the operators T_j^λ with λ satisfying the assumptions in (a) to $T_{\delta,j}$ can be estimated by

$$\|T_{\delta,j}^V\|_{p_c \rightarrow p'_c} \lesssim \sum_{\lambda_3=2}^{2^{2j}} \sum_{\lambda_1=2}^{2^j \lambda_3^{1/2}} \lambda_1^{\frac{3\theta-1}{2}} \lambda_3^{\frac{2\theta-2}{2}} \lesssim 1.$$

(b) **Assume next that** $\lambda_1 > 2^j \lambda_3^{1/2}$. Then $\lambda_3 \sim 1$ and $\lambda_1 \geq 2^j$, and thus

$$\|T_j^\lambda\|_{p_c \rightarrow p'_c} \lesssim 2^{(1-\theta)j} \lambda_1^{\frac{\theta-1}{2}}.$$

We thus find that the contributions $T_{\delta,j}^{VI}$ of the operators T_j^λ with λ satisfying the assumptions in (b) to $T_{\delta,j}$ can be estimated by

$$\|T_{\delta,j}^{VI}\|_{p_c \rightarrow p'_c} \lesssim \sum_{\lambda_1=2^j}^{\infty} 2^{(1-\theta)j} \lambda_1^{\frac{\theta-1}{2}} \lesssim 1.$$

As before, we see that

$$(5.48) \quad \|T_{\delta,j}^V\|_{p_c \rightarrow p'_c} + \|T_{\delta,j}^{VI}\|_{p_c \rightarrow p'_c} \lesssim 2^{(1-\frac{4}{p'_c})j}.$$

What is left is

4. Case: $\lambda_1 \sim \lambda_2 \sim \lambda_3$. We can here first apply the method of stationary phase to the integration in x_2 . This produces a phase function in x_1 , which is of the form $\phi_1(x_1) = \xi_1 x_1 + \xi_2(\rho(0)x_1^m + \text{error}) + \xi_3(\sigma\beta(0)x_1^n + \text{error})$, with small error terms of order $O(|\delta| + 2^{-j})$. We assume again that (5.19) holds true. Then ϕ_1 has a singularity of Airy type, which implies that the oscillatory integral with phase ϕ_1 that we have arrived at decays of order $O(|\lambda|^{-1/3})$. Indeed, we have $n \geq 2m + 1$ and $m \geq 2$, and since $x_1 \sim 1$, it is easy to see by studying the linear system of equations $y_j = \phi_1^{(j)}(x_1)$, $j = 1, 2, 3$, that there exist constants $0 < c_1 \leq c_2$ which do not depend on ξ and $x_1 \sim 1$ such that

$$c_1 |\xi| \leq \sum_{j=1}^3 |\phi_1^{(j)}(x_1)| \leq c_2 |\xi|.$$

Thus, our claim follows from Lemma 2.2. We thus find that

$$\|\widehat{\nu_j^\lambda}\|_\infty \lesssim \lambda_1^{-5/6}.$$

(a) **Assume first that** $\lambda_1 > 2^{2j}$. Then, by (5.43) $\|\nu_j^\lambda\|_\infty \lesssim 2^j \lambda_1$, and we obtain

$$\|T_j^\lambda\|_{p_c \rightarrow p'_c} \lesssim 2^{\theta j} \lambda_1^{\frac{11\theta-5}{6}}.$$

Lemma 5.2 shows that $11\theta - 5 < 0$, which implies that the contributions $T_{\delta,j}^{VII}$ of the operators T_j^λ with λ satisfying the assumptions in (a) to $T_{\delta,j}$ can again be estimated by

$$(5.49) \quad \|T_{\delta,j}^{VII}\|_{p_c \rightarrow p'_c} \lesssim \sum_{\lambda_1=2^{2j}}^{\infty} 2^{\theta j} \lambda_1^{\frac{11\theta-5}{6}} \lesssim 2^{\frac{14\theta-5}{3}j} \lesssim 2^{(1-\frac{4}{p'_c})j},$$

provided that either ϕ is of type D , or of type A and $m \geq 3$.

Observe also that if $m = 2$, then $\theta = 3/7$ and $p'_c = 14/3$, so that $\|T_j^\lambda\|_{p_c \rightarrow p'_c} \lesssim 2^{3j/7} \lambda_1^{-1/21}$, and

$$\sum_{\lambda_1 > 2^{6j}} 2^{\frac{3j}{7}} \lambda_1^{-\frac{1}{21}} \lesssim 2^{\frac{j}{7}} = 2^{(1-\frac{4}{p'_c})j},$$

so that in this case there only remain the terms with $\lambda_1 \leq 2^{6j}$.

(b) Assume finally that $\lambda_1 \leq 2^{2j}$. By the discussion in the preceding case, we may also assume that $m \geq 3$ when ϕ is of type A .

Then, by (5.43) $\|\nu_j^\lambda\|_\infty \lesssim \lambda_1^{3/2}$, and we obtain

$$\|T_j^\lambda\|_{p_c \rightarrow p'_c} \lesssim \lambda_1^{\frac{14\theta-5}{6}}.$$

We thus find that the contributions $T_{\delta,j}^{VIII}$ of the operators T_j^λ with λ satisfying the assumptions in (b) to $T_{\delta,j}$ can be estimated by

$$\|T_{\delta,j}^{VIII}\|_{p_c \rightarrow p'_c} \lesssim \sum_{\lambda_1=2}^{2^{2j}} \lambda_1^{\frac{14\theta-5}{6}}.$$

If $14\theta - 5 \leq 0$, then we immediately obtain the desired estimate $\|T_{\delta,j}^{VIII}\|_{p_c \rightarrow p'_c} \lesssim j \lesssim 2^{(1-\frac{4}{p'_c})j}$, so assume that $14\theta - 5 > 0$ (which means that $m < 9$ when ϕ is of type A). Then

$$\|T_{\delta,j}^{VIII}\|_{p_c \rightarrow p'_c} \lesssim 2^{\frac{14\theta-5}{3}j}.$$

If ϕ is of type D , then Lemma 5.2 shows that we again obtain the desired estimate, and similarly, if ϕ is of type A and $3 \leq m < 9$. We have thus shown that the estimate

$$(5.50) \quad \|T_{\delta,j}^{VIII}\|_{p_c \rightarrow p'_c} \lesssim 2^{(1-\frac{4}{p'_c})j},$$

holds true when either ϕ is of type D , or of type A_{n-1} with finite n and $m \geq 3$, or of type A_∞ (notice that in the latter case $\delta_0 = 0$, so the last sub-case $2^{2j}\delta_0 \sim 1$ does not appear).

The estimates (5.43) - (5.49) show that estimate (5.30) holds true also in this sub-case, which completes the proof of Proposition 5.1, with the exception of the case where ϕ is of type A_{n-1} with finite $n \geq 5$ and $m = 2$, in which we are left with the terms with $\lambda_1 \leq 2^{6j}$. Thus, in order to complete the proof of Proposition 5.1, and hence that of Theorem 1.7 when $h_{\text{lin}}(\phi) < 2$, it remains to prove the following

Proposition 5.3. *If ϕ is of type A_{n-1} , with $m = 2$ and finite $n \geq 5$, so that $p'_c = 14/3$ and $1 - 4/p'_c = 1/7$, then*

$$(5.51) \quad \left\| \sum_{2 \leq \lambda_1 \sim \lambda_2 \sim \lambda_3 \leq 2^{6j}} T_j^\lambda \right\|_{\frac{14}{11} \rightarrow \frac{14}{3}} \leq C 2^{\frac{j}{7}},$$

provided $j \in \mathbb{N}$ is sufficiently big and δ sufficiently small, where the constant C does neither depend on δ , nor on j .

The proof of this proposition, which will be given in the next section, requires a more refined estimation, making use of the fact that $\widehat{\nu_j^\lambda}(\xi)$ will be large only on a small neighborhood of some “Airy cone.”

6. PROOF OF PROPOSITION 5.3: AIRY TYPE ANALYSIS

In order to prove estimate (5.51), we shall assume for the sake of simplicity that

$$2 \leq \lambda_1 = \lambda_2 = \lambda_3 \leq 2^{6j}$$

(the remaining cases can be treated in a similar way). Recall also that $\sigma := 2^{2j}\delta_0 \sim 1$. In order to defray the notation, we shall in the sequel denote by λ the common value of $\lambda_1 = \lambda_2 = \lambda_3$, and put

$$\delta_3 := 2^{-j} \ll 1, \quad s_1 := \frac{\xi_1}{\xi_3}, \quad s_2 := \frac{\xi_2}{\xi_3},$$

so that $|s_1| \sim |s_2| \sim 1$. Moreover, we augment $\delta = (\delta_1, \delta_2)$ by the parameter δ_3 , and re-write

$$(6.1) \quad \Phi(x, \delta_1, \delta_2, j, \xi) = \xi_3 \tilde{\Phi}(x, \delta, \sigma, s_1, s_2),$$

where now $\delta := (\delta_1, \delta_2, \delta_3)$ and

$$\begin{aligned} \tilde{\Phi}(x, \delta, \sigma, s_1, s_2) &:= s_1 x_1 + s_2 x_1^2 \omega(\delta_1 x_1) + \sigma x_1^n \beta(\delta_1 x_1) \\ &\quad + \delta_3 s_2 x_2 + x_2^2 b_0(x, \delta), \end{aligned}$$

and where (compare (5.5))

$$b_0(x, \delta) := b(\delta_1 x_1, \delta_2 \delta_3 x_2 + \delta_2 x_1^m \omega(\delta_1 x_1)).$$

Notice that

$$b_0(x, 0) = b(0, 0) \neq 0.$$

We denote accordingly the measure ν_j^λ by ν_δ^λ (which indeed also depends on $\sigma \sim 1$, but we shall suppress this dependency in order to defray the notation), i.e.,

$$\widehat{\nu_\delta^\lambda}(\xi) := \chi_1\left(\frac{s_1 \xi_3}{\lambda}\right) \chi_1\left(\frac{s_2 \xi_3}{\lambda}\right) \chi_1\left(\frac{\xi_3}{\lambda}\right) \int e^{-i \xi_3 \tilde{\Phi}(x, \delta, \sigma, s_1, s_2)} \tilde{a}(x, \delta) dx,$$

where the amplitude $\tilde{a}(x, \delta)$ is a smooth function of x supported where $x_1 \sim 1$ and $|x_2| \lesssim 1$, whose derivatives are uniformly bounded with respect to the parameters δ . Moreover, if T_δ^λ denotes the convolution operator

$$T_\delta^\lambda \varphi := \varphi * \widehat{\nu_\delta^\lambda},$$

then we see that the estimate (5.51) can be re-written as

$$(6.2) \quad \left\| \sum_{2 \leq \lambda \leq \delta_3^{-6}} T_\delta^\lambda \right\|_{\frac{14}{11} \rightarrow \frac{14}{3}} \leq C \delta_3^{-\frac{1}{7}}$$

We shall need to understand the precise behavior of $\widehat{\nu_\delta^\lambda}(\xi)$. To this end, consider the integration with respect to x_2 in the corresponding integral. Depending on the signs of s_2 and b_0 , there may be a critical point, or not. If there is no critical point, then we may integrate by parts in x_2 , which leads again to favorable estimates $\|\widehat{\nu_\delta^\lambda}\|_\infty = O(\lambda^{-N})$.

We shall thus assume that the relative signs are so that there is a critical point x_2^c . Writing $x_2 = \delta_3 s_2 y_2$, and applying the implicit function theorem to y_2 , we find that

$$(6.3) \quad x_2^c = \delta_3 s_2 Y_2(\delta_1 x_1, \delta, s_2),$$

where Y_2 is smooth and of size $|Y_2| \sim 1$. Notice also that $Y_2(0, 0, s_2) = -1/(2b(0, 0))$ when $\delta = 0$. Let us put

$$(6.4) \quad \begin{aligned} \Psi(x_1, \delta, \sigma, s_1, s_2) &:= \tilde{\Phi}(x_1, x_2^c, \sigma, s_1, s_2) \\ &= s_1 x_1 + s_2 x_1^2 \omega(\delta_1 x_1) + \sigma x_1^n \beta(\delta_1 x_1) + \delta_3 s_2 x_2^c + (x_2^c)^2 b_0(x_1, x_2^c, \delta). \end{aligned}$$

Let us first consider the case where $\delta = 0$. Then

$$\Psi(x_1, 0, \sigma, s_1, s_2) := s_1 x_1 + s_2 x_1^2 \omega(0) + \sigma x_1^n \beta(0),$$

and depending again on the signs of $s_2 \omega(0)$ and $\beta(0)$, the first derivative Ψ' (with respect to x_1) may have a critical point, or not. If not, Ψ will at most have non-degenerate critical points, and this case can be treated again by the method of stationary phase, respectively integrations by parts. We shall therefore concentrate on the case where Ψ' does have a critical point x_1^c , which will then be given explicitly by

$$x_1^c = x_1^c(0, \sigma, s_2) := \left(-\frac{2\omega(0)}{n(n-1)\sigma\beta(0)} s_2 \right)^{\frac{1}{n-2}}.$$

Let us assume that $s_2 > 0$ (the case where it is negative can be treated similarly). By scaling in x_1 , we may and shall assume for simplicity that

$$(6.5) \quad -\frac{2\omega(0)}{n(n-1)\sigma\beta(0)} = 1 \quad (\text{and } s_2 \sim 1).$$

Then $x_1^c(0, \sigma, s_2) = s_2^{\frac{1}{n-2}}$, and $|\Psi'''(x_1^c, 0, \sigma, s_1, s_2)| \sim 1$. Thus, the implicit function theorem shows that for δ sufficiently small, there is a unique critical point $x_1^c = x_1^c(\delta, \sigma, s_2)$ of Ψ' depending smoothly on δ, σ and s_2 , i.e.,

$$(6.6) \quad \Psi''(x_1^c(\delta, \sigma, s_2), \delta, \sigma, s_1, s_2) = 0.$$

Let us write $s := (s_1, s_2)$.

Lemma 6.1. *The phase Ψ given by (6.4) can be developed locally around the critical point x_1^c of Ψ' in the form*

$$\Psi(x_1^c(\delta, \sigma, s_2) + y_1, \delta, \sigma, s_1, s_2) = B_0(s, \delta, \sigma) - B_1(s, \delta, \sigma)y_1 + B_3(s_2, \delta, \sigma, y_1)y_1^3,$$

where B_0, B_1 and B_3 are smooth functions, and where $|B_3(s_2, \delta, \sigma, y_1)| \sim 1$, and indeed

$$B_3(s_2, \delta, \sigma, 0) = s_2^{\frac{n-3}{n-2}} G_4(s_2, \delta, \sigma),$$

where

$$G_4(s_2, 0, \sigma) = \frac{n(n-1)(n-2)}{6} \sigma \beta(0).$$

Moreover, we may write

$$(6.7) \quad \begin{cases} x_1^c(\delta, \sigma, s_2) &= s_2^{\frac{1}{n-2}} G_1(s_2, \delta, \sigma), \\ B_0(s, \delta, \sigma) &= s_1 s_2^{\frac{1}{n-2}} G_1(s_2, \delta, \sigma) - s_2^{\frac{n}{n-2}} G_2(s_2, \delta, \sigma), \\ B_1(s, \delta, \sigma) &= -s_1 + s_2^{\frac{n-1}{n-2}} G_3(s_2, \delta, \sigma), \end{cases}$$

where

$$(6.8) \quad \begin{cases} G_1(s_2, 0, \sigma) &= 1, \\ G_2(s_2, 0, \sigma) &= \frac{n^2-n-2}{2} \sigma \beta(0), \\ G_3(s_2, 0, \sigma) &= n(n-2) \sigma \beta(0). \end{cases}$$

Notice that all the numbers in (6.8) are non-zero, since we assume $n \geq 5$.

Proof. The first statements in (6.7), (6.8) are obvious. Next, by (6.4) and (6.3) we have

$$\begin{aligned} B_0(s, \delta, \sigma) &= \Psi(x_1^c(\delta, \sigma, s_2), \delta, \sigma, s_1, s_2) = s_1 s_2^{\frac{1}{n-2}} G_1(s_2, \delta, \sigma) \\ &+ s_2^{\frac{n}{n-2}} \left(G_1(s_2, \delta, \sigma)^2 \omega(\delta_1 x_1^c) + \sigma G_1(s_2, \delta, \sigma)^n \beta(\delta_1 x_1^c) + \right. \\ &\left. + \delta_3^2 s_2^{\frac{n-4}{n-2}} (Y_2(\delta_1 x_1^c, \delta, s_2) + Y_2(\delta_1 x_1^c, \delta, s_2)^2 b_0(x_1^c, \delta)) \right), \end{aligned}$$

where x_1^c is given by the first identity in (6.7). In combination with (6.5), we thus obtain the second identity in (6.7) and the third in (6.8), because $s_2 \sim 1$.

Similarly,

$$\begin{aligned} -B_1(s, \delta, \sigma) &= \Psi'(x_1^c(\delta, \sigma, s_2), \delta, \sigma, s_1, s_2) \\ &= s_1 + 2s_2 x_1^c \omega(\delta_1 x_1^c) + n \sigma (x_1^c)^{n-1} \beta(\delta_1 x_1^c) + O(|\delta|), \end{aligned}$$

which in view of (6.5) easily implies the last identities in (6.7) and (6.8). Finally, when $y_1 = 0$, then

$$6B_3(s_2, \delta, \sigma, 0) = \Psi'''(x_1^c(\delta, \sigma, s_2), \delta, \sigma, s_1, s_2) = n(n-1)(n-2) \sigma \beta(0) (x_1^c)^{n-3} + O(|\delta|),$$

which shows that $|B_3(s_2, \delta, \sigma, y_1)| \sim 1$ for $|y_1|$ sufficiently small.

Q.E.D.

Applying the method of stationary phase with parameters to the x_2 -integration and Lemma 6.1, and ignoring the region away from the critical point x_1^c , which leads to better estimates by means of integrations by parts, we find that we may assume that

$$\begin{aligned}
\widehat{\nu_\delta^\lambda}(\xi) &= |\xi_3|^{-1/2} \chi_1\left(\frac{s_1 \xi_3}{\lambda}\right) \chi_1\left(\frac{s_2 \xi_3}{\lambda}\right) \chi_1\left(\frac{\xi_3}{\lambda}\right) e^{-i\xi_3 B_0(s, \delta, \sigma)} \\
(6.9) \quad &\int e^{-i\xi_3 \left(B_3(s_2, \delta, \sigma, y_1) y_1^3 - B_1(s, \delta, \sigma) y_1 \right)} a(y_1, \delta, \sigma, s, \xi_3) \chi_0(y_1) dy_1,
\end{aligned}$$

where $a(x_1, \delta, \sigma, s, \xi_3)$ is smooth and a symbol of order 0 with respect to ξ_3 , uniformly in x_1, δ, σ, s . Moreover, χ_0 is a smooth cut-off function supported in sufficiently small neighborhood of the origin.

We shall here make use of the following, more or less classical lemma (compare for instance Lemma 1 in [18], or [8] for related results) in the case of Airy type integrals, i.e., when $B = 3$. The case of general $B \geq 3$ will become relevant in [15]. Since we need somewhat more refined results than what can be found in the literature, for instance information of the asymptotic behavior also under certain perturbations, we shall sketch a proof.

Lemma 6.2. *Let $B \geq 3$ be an integer, and let*

$$J(\lambda, u, s) := \int e^{i\lambda(b(t,s)t^B - ut - \sum_{j=2}^{B-1} b_j(u)t^j)} a(t, s) dt, \quad \lambda \geq 1,$$

where a, b and the b_j are smooth, real-valued functions on an open neighborhood of $I \times K$, where I is a compact neighborhood of the origin in \mathbb{R} and K is a compact subset of \mathbb{R}^m . Assume also that $b(t, s) \neq 0$ on $I \times K$, that $|t| \leq \varepsilon$ on the support of a , and that $|u| \lesssim 1$. We also assume that

$$|b_j(u)| \leq C|u|, \quad j = 2, \dots, B-1.$$

If $\varepsilon > 0$ is chosen sufficiently small and λ sufficiently large, then the following hold true:

(a) *If $\lambda^{(B-1)/B}|u| \lesssim 1$, then*

$$J(\lambda, u, s) = \lambda^{-\frac{1}{B}} g(\lambda^{\frac{B-1}{B}} u, \lambda, s),$$

where $g(v, \lambda, s)$ is a smooth function of (v, λ, s) whose derivatives of any order are uniformly bounded on its natural domain.

(b) *If $\lambda^{(B-1)/B}|u| \gg 1$, let us assume first that u and b have the same sign, and that B is odd. Then*

$$\begin{aligned}
&J(\lambda, u, s) \\
&= \lambda^{-\frac{1}{2}} |u|^{-\frac{B-2}{2B-2}} \left(a_+(|u|^{\frac{1}{B-1}}, s) e^{i\lambda|u|^{\frac{B}{B-1}} q_+(|u|^{\frac{1}{B-1}}, s)} + a_- (|u|^{\frac{1}{B-1}}, s) e^{i\lambda|u|^{\frac{B}{B-1}} q_- (|u|^{\frac{1}{B-1}}, s)} \right) \\
&\quad + (\lambda|u|)^{-1} E(\lambda|u|^{\frac{B}{B-1}}, |u|^{\frac{1}{B-1}}, \lambda, s),
\end{aligned}$$

where a_\pm, q_\pm are smooth functions, and where E is smooth and satisfies estimates

$$|\partial_\mu^\alpha \partial_v^\beta \partial_s^\gamma E(\mu, v, \lambda, s)| \leq C_{N, \alpha, \beta, \gamma} |v|^{-\beta} |\mu|^{-N}, \quad N, \alpha, \beta, \gamma \in \mathbb{N}.$$

Moreover, when $|u|$ is sufficiently small, then

$$q_{\pm}(v, s) = \mp \operatorname{sgn} b(0, s) |b(0, s)|^{\frac{1}{B-1}} \rho(v, s),$$

where ρ is smooth and $\rho(0, s) = (B-1) \cdot B^{-B/(B-1)}$.

Finally, if u and b have opposite signs, then the same formula remains valid, even with $a_+ \equiv 0, a_- \equiv 0$. And, if B is even, we do have a similar result, but without the presence of the term containing a_- .

Proof. In the case (a), scaling in t by the factor $\lambda^{-1/B}$ allows to re-write

$$J(\lambda, u, s) = \lambda^{-\frac{1}{B}} \int e^{i(b(\lambda^{-\frac{1}{B}}t, s)t^B - \lambda^{\frac{B-1}{B}}ut - \sum_{j=2}^{B-1} \lambda^{\frac{B-j}{B}} b_j(u)t^j)} a(\lambda^{-\frac{1}{B}}t, s) dt.$$

Choose a smooth cut-off function χ_0 on \mathbb{R} which is identically one on $[-1, 1]$, and $M \gg 1$, and decompose

$$\lambda^{\frac{1}{B}} J(\lambda, u, s) = G_0(\lambda^{\frac{B-1}{B}}u, \lambda, s) + G_{\infty}(\lambda^{\frac{B-1}{B}}u, \lambda, s),$$

where, for $|v| \lesssim 1$,

$$G_0(v, \lambda, s) := \int e^{i(b(\lambda^{-\frac{1}{B}}t, s)t^B - vt - \sum_{j=2}^{B-1} \lambda^{\frac{B-j}{B}} b_j(\lambda^{\frac{1-B}{B}}v)t^j)} \chi_0\left(\frac{t}{M}\right) a(\lambda^{-\frac{1}{B}}t, s) dt,$$

$$G_{\infty}(v, \lambda, s) := \int e^{i(b(\lambda^{-\frac{1}{B}}t, s)t^B - vt - \sum_{j=2}^{B-1} \lambda^{\frac{B-j}{B}} b_j(\lambda^{\frac{1-B}{B}}v)t^j)} (1 - \chi_0\left(\frac{t}{M}\right)) a(\lambda^{-\frac{1}{B}}t, s) dt.$$

Notice that for $j \geq 2$,

$$|\lambda^{\frac{B-j}{B}} b_j(\lambda^{\frac{1-B}{B}}v)| \leq C \lambda^{\frac{B-j}{B}} \lambda^{\frac{1-B}{B}} |v| \lesssim \lambda^{-1}.$$

It is then easy to see that G_0 is a smooth function of (v, λ, s) whose derivatives of any order are uniformly bounded on its natural domain, and the same can easily be verified for G_{∞} by means of iterated integrations by parts. This proves (a).

In order to prove (b), consider first the case where $|u| \geq \varepsilon$. If $\Phi = \Phi(t)$ denotes the complete phase in the oscillatory integral defining $J(\lambda, u, s)$, recalling that $|t| \leq \varepsilon$, we easily see that

$$|\Phi'(t)| \geq C\lambda|u|,$$

provided we choose ε' and ε sufficiently small. Integrations by parts then show that we can represent $J(\lambda, u, s)$ by the third term $(\lambda|u|)^{-1} E(\lambda|u|^{\frac{B}{B-1}}, |u|^{\frac{1}{B-1}}, \lambda, s)$.

Let us therefore assume that $|u| < \varepsilon$. We shall also assume that $u > 0$; the case $u < 0$ can be treated in a similar way. Here, we scale t by the factor $u^{1/B-1}$, and re-write

$$J(\lambda, u, s) = u^{\frac{1}{B-1}} \int e^{i\lambda u^{\frac{B}{B-1}} (b(u^{\frac{1}{B-1}}t, s)t^B - t - \sum_{j=2}^{B-1} u^{-\frac{B-j}{B-1}} b_j(u)t^j)} a(u^{\frac{1}{B-1}}t, s) dt.$$

Again, we decompose this as

$$J(\lambda, u, s) = J_0(\lambda, u^{\frac{1}{B-1}}, s) + J_{\infty}(\lambda, u^{\frac{1}{B-1}}, s),$$

where, with $v := u^{\frac{1}{B-1}}$,

$$\begin{aligned} J_0(\lambda, v, s) &:= v \int e^{i\lambda v^B(b(vt, s)t^B - t - \sum_{j=2}^{B-1} v^{-(B-j)} b_j(v^{B-1})t^j)} \chi_0\left(\frac{t}{M}\right) a(vt, s) dt, \\ J_\infty(\lambda, v, s) &:= v \int e^{i\lambda v^B(b(vt, s)t^B - t - \sum_{j=2}^{B-1} v^{-(B-j)} b_j(v^{B-1})t^j)} (1 - \chi_0\left(\frac{t}{M}\right)) a(vt, s) dt. \end{aligned}$$

Observe that

$$|v^{-(B-j)} b_j(v^{B-1})| \leq C v^{j-1} \lesssim \varepsilon^{\frac{1}{B-1}}, \quad j = 2, \dots, B-1.$$

Assume that ε is sufficiently small. If B is odd, then, in the first integral J_0 , the phase has exactly two non-degenerate critical points $t_\pm(v, s) \sim \pm 1$, if $b > 0$, and thus the method of stationary phase shows that

$$J_0(\lambda, v, s) = v(\lambda v^B)^{-\frac{1}{2}} a_+(v, s) e^{i\lambda v^B q_+(v, s)} + v(\lambda v^B)^{-\frac{1}{2}} a_-(v, s) e^{i\lambda v^B q_-(v, s)} + v E_1(\lambda v^B, v, \lambda, s),$$

where a_\pm are smooth functions, and where E_1 is smooth and rapidly decaying with respect to the first variable. If $b < 0$, then there are no critical points, and we get the term E_1 only. Moreover,

$$q_\pm(v, s) = b(vt_\pm(v, s), s)t_\pm(v, s)^B - t_\pm(v, s) + O(v).$$

Note that if $v = 0$, then $t_\pm(0, s) = \pm(Bb(0, s))^{-1/(B-1)}$, so that

$$q_\pm(0, s) = \mp \frac{B-1}{B^{\frac{B}{B-1}} b(0, s)^{\frac{1}{B-1}}} \neq 0,$$

which proves the statement about q_\pm . A similar discussion applies when B is even. In this case, there is only the one critical point, namely $t_+(v, s)$.

In the second integral J_∞ , we may apply integrations by parts in order to re-write it as

$$J_\infty(\lambda, v, s) := v(\lambda v^B)^{-N} \int e^{i\lambda v^B(b(vt, s)t^B - t - \sum_{j=2}^{B-1} v^{-(B-j)} b_j(v^{B-1})t^j)} a_N(t, v, \lambda, s) dt, \quad N \in \mathbb{N},$$

where a_N is supported where $|t| \geq M$ and $|a_N(t, v, \lambda, s)| \leq C_N |t|^{-2N}$. Similarly, if we take derivatives with respect to s , we produce additional powers of t in the integrand, which, however, can be compensated by integrations by parts. Analogous considerations apply to derivatives with respect to v (where we produce negative powers of v), and with respect to λv^3 . Altogether, we find that

$$J_\infty(\lambda, v, s) = \frac{1}{\lambda v^{B-1}} E_2(\lambda v^B, v, \lambda, s),$$

where E_2 is smooth and

$$|\partial_\mu^\alpha \partial_v^\beta \partial_s^\gamma E_2(\mu, v, \lambda, s)| \leq C_{N, \alpha, \beta, \gamma} |v|^{-\beta} |\mu|^{-N}, \quad N, \alpha, \beta, \gamma \in \mathbb{N}.$$

Summing up all terms, and putting $E := E_1 + E_2$, we obtain the statements in (b).

Q.E.D.

The following remark can be verified easily by the well-known versions of the method of stationary phase for oscillatory integrals whose amplitude depends also on the parameter λ as symbols of order 0 (see, e.g., [20]).

Remark 6.3. *We may even allow in Lemma 6.2 that the function $a(t, s)$ also depends on λ , i.e., $a = a(t, s, \lambda)$, in such a way that it is a symbol of order 0 in λ , uniformly in the other parameters, i.e.,*

$$|(\frac{\partial}{\partial \lambda})^\alpha (\frac{\partial}{\partial t})^{\beta_1} (\frac{\partial}{\partial s})^{\beta_2} a(t, s, \lambda)| \leq C_{\alpha, \beta} (1 + \lambda)^{-\alpha}$$

for all $\alpha, \beta_1, \beta_2 \in \mathbb{N}$. Then the same conclusions hold, only with a_\pm and E depending also additionally on λ as symbols of order 0 in a uniform way.

Let us apply this lemma and the remark to the oscillatory integral (6.9), with $B = 3$. Putting $u := B_1(s, \delta, \sigma)$, in view of this lemma we shall decompose the frequency support of ν_δ^λ furthermore into the domain where $\lambda^{2/3} |B_1(s, \delta, \sigma)| \lesssim 1$ (this is essentially a conic region in ξ -space, which will be called the "region near the Airy cone", where the *Airy cone* is given by $B_1 = 0$, i.e.,

$$s_1 = s_2^{\frac{n-1}{n-2}} G_3(s_2, \delta, \sigma)$$

), and the remaining domain into the conic regions where $(2^{-l}\lambda)^{2/3} |B_1(s, \delta, \sigma)| \sim 1$, for $M_0 \leq 2^l \leq \frac{\lambda}{M_1}$ where $M_0, M_1 \in \mathbb{N}$ are sufficiently large. To this end, we choose smooth cut-off functions χ_0 and χ_1 such that χ_0 has sufficiently small support and $\chi_0 = 1$ on a neighborhood of the origin, and $\chi_1(t)$ is supported where $|t| \sim 1$ and $\sum_{l \in \mathbb{Z}} \chi_1(2^{-2l/3}) = 1$ on $\mathbb{R} \setminus \{0\}$, and define the functions $\nu_{\delta, Ai}^\lambda$ and $\nu_{\delta, l}^\lambda$ by

$$\begin{aligned} \widehat{\nu_{\delta, Ai}^\lambda}(\xi) &:= \chi_0(\lambda^{2/3} B_1(s, \delta, \sigma)) \widehat{\nu_\delta^\lambda}(\xi), \\ \widehat{\nu_{\delta, l}^\lambda}(\xi) &:= \chi_1((2^{-l}\lambda)^{2/3} B_1(s, \delta, \sigma)) \widehat{\nu_\delta^\lambda}(\xi), \quad M_0 \leq 2^l \leq \frac{\lambda}{M_1}, \end{aligned}$$

so that

$$(6.10) \quad \nu_\delta^\lambda = \nu_{\delta, Ai}^\lambda + \sum_{M_0 \leq 2^l \leq \frac{\lambda}{M_1}} \nu_{\delta, l}^\lambda.$$

Denote by $T_{\delta, Ai}^\lambda$ and $T_{\delta, l}^\lambda$ the convolution operators

$$T_{\delta, Ai}^\lambda \varphi := \varphi * \widehat{\nu_{\delta, Ai}^\lambda}, \quad T_{\delta, l}^\lambda \varphi := \varphi * \widehat{\nu_{\delta, l}^\lambda}.$$

6.1. Estimation of $T_{\delta, Ai}^\lambda$. We first consider the region near the Airy cone and prove the following

Lemma 6.4. *There are constants C_1, C_2 so that*

$$(6.11) \quad \|\widehat{\nu_{\delta, Ai}^\lambda}\|_\infty \leq C_1 \lambda^{-\frac{5}{6}},$$

$$(6.12) \quad \|\nu_{\delta, Ai}^\lambda\|_\infty \leq C_2 \lambda^{\frac{7}{6}},$$

uniformly in s and σ , provided λ is sufficiently large and δ sufficiently small.

Notice that by interpolation (again with $\theta = 3/7$) these estimates imply that

$$\|T_{\delta,Ai}^\lambda\|_{14/11 \rightarrow 14/3} \lesssim (\lambda^{-\frac{5}{6}})^{\frac{4}{7}} (\lambda^{\frac{7}{6}})^{\frac{3}{7}} = \lambda^{\frac{1}{42}},$$

so that

$$(6.13) \quad \sum_{2 \leq \lambda \leq \delta_3^{-6}} \|T_{\delta,Ai}^\lambda\|_{14/11 \rightarrow 14/3} \lesssim \delta_3^{-\frac{1}{7}},$$

which is exactly the estimate that we need (cf. (6.2)).

Let us turn to the proof of Lemma 6.4. The first estimate (6.11) is immediate from (6.9) and Lemma 6.2.

In order to prove the second estimate, observe first that by Lemma 6.2 (a) and the subsequent remark, we may write

$$\begin{aligned} \chi_0(\lambda^{2/3} B_1(s, \delta, \sigma)) \int e^{-i\xi_3 \left(B_3(s_2, \delta, \sigma, y_1) y_1^3 - B_1(s, \delta, \sigma) y_1 \right)} a(y_1, \delta, \sigma, s, \xi_3) \chi_0(y_1) dy_1 \\ = |\xi_3|^{-\frac{1}{3}} \chi_0(\lambda^{2/3} B_1(s, \delta, \sigma)) g\left(|\xi_3|^{2/3} |B_1(s, \delta, \sigma)|, \xi_3, \delta, \sigma, s\right), \end{aligned}$$

where g is a smooth function whose derivatives of any order are uniformly bounded on its natural domain.

Applying the Fourier inversion formula to $\nu_{\delta,Ai}^\lambda$, (6.9) and this identity yield that

$$\begin{aligned} \nu_{\delta,Ai}^\lambda(x) = \iint |\xi_3|^{-\frac{1}{3}} \sigma_{-1/2}(\xi) \chi_0(\lambda^{2/3} B_1(s, \delta, \sigma)) \chi_1\left(\frac{s_1 \xi_3}{\lambda}\right) \chi_1\left(\frac{s_2 \xi_3}{\lambda}\right) \chi_1\left(\frac{\xi_3}{\lambda}\right) e^{i\xi \cdot x} \\ e^{-i\xi_3 B_0(s, \delta, \sigma)} g\left(|\xi_3|^{2/3} B_1(s, \delta, \sigma), \xi_3, \delta, \sigma, s\right) d\xi. \end{aligned}$$

We change coordinates from $\xi = (\xi_1, \xi_2, \xi_3)$ to s_1, s_2 and $s_3 := \xi_3/\lambda$, i.e.,

$$\xi_1 = s_1 \xi_3 = \lambda s_1 s_3, \quad \xi_2 = \lambda s_2 \xi_3 = \lambda s_2 s_3, \quad \xi_3 = \lambda s_3,$$

and write in the sequel

$$s := (s_1, s_2, s_3) \quad s' := (s_1, s_2).$$

Observing also that the functions

$$\sigma_\lambda(s) := \lambda^{\frac{1}{2}} \sigma_{-1/2}(\lambda s_1 s_3, \lambda s_2 s_3, \lambda s_3)$$

form a family of smooth functions whose C^N -norms on the domain where $s_j \sim 1$, $j = 1, 2, 3$, are uniformly bounded in λ (and δ, σ), and which, as functions of λ , are symbols of order 0, uniformly with respect to s, δ, σ .

We then find that

$$(6.14) \quad \begin{aligned} \nu_{\delta,Ai}^\lambda(x) &= \lambda^{\frac{13}{6}} \int e^{-i\lambda s_3 \left(B_0(s', \delta, \sigma) - s_1 x_1 - s_2 x_2 - x_3 \right)} \chi_0(\lambda^{2/3} B_1(s', \delta, \sigma)) \\ &\quad \sigma_\lambda(s) g\left(\lambda^{2/3} B_1(s', \delta, \sigma), \lambda, \delta, \sigma, s\right) \tilde{\chi}(s) ds_1 ds_2 ds_3, \end{aligned}$$

where

$$\tilde{\chi}(s) := \chi_1(s_1 s_3) \chi_1(s_2 s_3) \chi_1(s_3)$$

localizes to are region where $s_j \sim 1$, $j = 1, 2, 3$. The function g appearing here may not coincide with the one in previous formulas, but has similar properties.

Observe first that when $|x| \gg 1$, then we easily obtain by means of integrations by parts that

$$(6.15) \quad |\nu_{\delta, Ai}^\lambda(x)| \leq C_N \lambda^{-N}, \quad N \in \mathbb{N}, \text{ if } |x| \gg 1.$$

Indeed, when $|x_1| \gg 1$, then we integrate by parts repeatedly in s_1 to see this, and a similar argument applies when $|x_2| \gg 1$, where we use the s_2 -integration. Observe that in each step, we gain a factor λ^{-1} , and lose at most $\lambda^{2/3}$. Finally, when $|x_1| + |x_2| \lesssim 1$ and $|x_3| \gg 1$, then we can integrate by parts in s_3 in order to establish this estimate.

We may therefore assume now that $|x| \lesssim 1$.

We then perform yet another change of coordinates, passing from $s' = (s_1, s_2)$ to (z, s_2) , where

$$z := \lambda^{2/3} B_1(s', \delta, \sigma).$$

Applying (6.7), we find that

$$z = \lambda^{2/3} (-s_1 + s_2^{\frac{n-1}{n-2}} G_3(s_2, \delta, \sigma))$$

so that

$$(6.16) \quad s_1 = s_2^{\frac{n-1}{n-2}} G_3(s_2, \delta, \sigma) - \lambda^{-2/3} z.$$

In combination with (6.7), we thus obtain that

$$(6.17) \quad B_0(s, \delta, \sigma) = -\lambda^{-2/3} z s_2^{\frac{1}{n-2}} G_1(s_2, \delta, \sigma) + s_2^{\frac{n}{n-2}} (G_1 G_3 - G_2)(s_2, \delta, \sigma).$$

We may thus re-write

$$(6.18) \quad \begin{aligned} \nu_{\delta, Ai}^\lambda(x) &= \lambda^{\frac{3}{2}} \int e^{-i\lambda s_3 \Phi(z, s_2, x_1, \delta, \sigma)} g\left(z, \lambda, \delta, \sigma, s_2^{\frac{n-1}{n-2}} G_3(s_2, \delta, \sigma) - \lambda^{-2/3} z, s_2, s_3\right) \\ &\quad (\sigma_\lambda \tilde{\chi})\left(s_2^{\frac{n-1}{n-2}} G_3(s_2, \delta, \sigma) - \lambda^{-2/3} z, s_2\right) \chi_0(z) dz ds_2 ds_3, \end{aligned}$$

where

$$(6.19) \quad \begin{aligned} \Phi(z, s_2, x_1, \delta, \sigma) &:= s_2^{\frac{n}{n-2}} (G_1 G_3 - G_2)(s_2, \delta, \sigma) - s_2^{\frac{n-1}{n-2}} G_3(s_2, \delta, \sigma) x_1 - s_2 x_2 - x_3 \\ &\quad + \lambda^{-2/3} z (x_1 - s_2^{\frac{1}{n-2}} G_1(s_2, \delta, \sigma)). \end{aligned}$$

Observe that by (6.7), when $\delta = 0$,

$$(6.20) \quad (G_1 G_3 - G_2)(s_2, \delta, \sigma) = \frac{n^2 - 3n + 2}{2} \sigma \beta(0) \neq 0, \quad G_3(s_2, \delta, \sigma) = n(n-2) \sigma \beta(0) \neq 0,$$

since we assume that $n \geq 5$, and that the exponents $n/(n-2)$, $(n-1)/(n-2)$ and 1 of s_2 which appear in Φ (regarding the last term in (6.19) as an error term) are all

different. Moreover, recall that $|x| \lesssim 1$. It is then easily seen that this implies that, when $\delta = 0$,

$$\sum_{j=1}^3 |\partial_{s_2}^j \Phi(z, s_2, x_1, \delta, \sigma)| \sim 1 \quad \text{for every } s_2 \sim 1,$$

uniformly in z and σ . The same type of estimates then remains valid for δ sufficiently small. We may thus apply the van der Corput type Lemma 2.2 to the s_2 -integration in (6.18), which in combination with Fubini's theorem yields

$$\|\nu_{\delta, Ai}^\lambda\|_\infty \leq C \lambda^{\frac{3}{2}} \lambda^{-\frac{1}{3}},$$

hence (6.12). This concludes the proof of Lemma 6.4.

6.2. Estimation of $T_{\delta, l}^\lambda$. We next regard the region away from the Airy cone and prove the following

Lemma 6.5. *There are constants C_1, C_2 so that*

$$(6.21) \quad \|\widehat{\nu_{\delta, l}^\lambda}\|_\infty \leq C_1 2^{-\frac{l}{6}} \lambda^{-\frac{5}{6}},$$

$$(6.22) \quad \|\nu_{\delta, l}^\lambda\|_\infty \leq C_2 \lambda^{\frac{7}{6}},$$

uniformly in s and σ , provided λ is sufficiently large and δ sufficiently small.

Notice that by interpolation (again with $\theta = 3/7$) these estimates imply that

$$\|T_{\delta, l}^\lambda\|_{14/11 \rightarrow 14/3} \lesssim (2^{-\frac{l}{6}} \lambda^{-\frac{5}{6}})^{\frac{4}{7}} (\lambda^{\frac{7}{6}})^{\frac{3}{7}} = 2^{-\frac{2l}{21}} \lambda^{\frac{1}{42}},$$

so that

$$(6.23) \quad \sum_{M_0 \leq 2^l \leq \frac{\lambda}{M_1}} \sum_{2 \leq \lambda \leq \delta_3^{-6}} \|T_{\delta, Ai}^\lambda\|_{14/11 \rightarrow 14/3} \lesssim \delta_3^{-\frac{1}{7}},$$

which is again exactly the estimate that we need in order to verify (6.2), and so the proof of Lemma 6.5 will also conclude the proof of Proposition 5.3.

Let us turn to the proof of Lemma 6.5. Applying Lemma 6.2 (b) to the integration with respect to y_1 in (6.9), and observing that here $|u| = |B_1(s_2, \delta, \sigma, y_1)| \sim (2^{-l} \lambda)^{-2/3}$, we obtain that

$$\|\widehat{\nu_{\delta, l}^\lambda}\|_\infty \leq C \lambda^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} \left((2^{-l} \lambda)^{-\frac{2}{3}} \right)^{-\frac{1}{4}},$$

which yields the first estimate (6.21).

In order to prove the second estimate in the lemma, we apply Lemma 6.2 (b) and the subsequent remark to

$$\int e^{-i\xi_3 \left(B_3(s_2, \delta, \sigma, y_1) y_1^3 - B_1(s, \delta, \sigma) y_1 \right)} a(y_1, \delta, \sigma, s, \xi_3) \chi_0(y_1) dy_1.$$

Since we assume that $\lambda^{-2/3} |B_1(s_2, \delta, \sigma, y_1)| \sim 2^{2l/3} \gg 1$, this allows to express this integral as a sum of at most three terms, and we shall concentrate on one of the first two terms appearing in Lemma 6.2 (b), since the remaining term leads to even more

favorable estimates. So, abusing notation slightly, let us assume for simplicity that the integral is given just by the first term, which leads to

$$\begin{aligned} & \int e^{-i\xi_3 \left(B_3(s_2, \delta, \sigma, y_1) y_1^3 - B_1(s, \delta, \sigma) y_1 \right)} a(y_1, \delta, \sigma, s) \chi_0(y_1) dy_1 \\ &= |\xi_3|^{-\frac{1}{2}} |B_1|^{-\frac{1}{4}} a_+ (|B_1|^{\frac{1}{2}}, s, \delta, \sigma) e^{-i|\xi_3| |B_1|^{\frac{3}{2}} q_+ (|B_1|^{\frac{1}{2}}, s, \delta, \sigma)}, \end{aligned}$$

where $B_1 = B_1(s, \delta, \sigma)$, and where a_+, q_+ are smooth, and where

$$(6.24) \quad q_+(v, s, \delta, \sigma) = -|B_3(s_2, \delta, \sigma, 0)|^{\frac{1}{2}} \operatorname{sgn} B_3(s_2, \delta, \sigma, 0) \rho(v, s, \delta, \sigma),$$

where ρ is smooth and $\rho(0, s, \delta, \sigma) = 2 \cdot 3^{-3/2}$.

Indeed, a_+ will also depend on ξ_3 as a symbol of order 0, but we shall suppress this dependence (which has no consequences for the arguments to follow) in the sequel in order to defray the notation.

Observe also that by Lemma 6.1, when $\delta = 0$, then

$$(6.25) \quad q_+(v, s, \delta, \sigma) = c_0 s_1^{\frac{n-3}{2(n-2)}} \rho(v, s, \delta, \sigma) \neq 0$$

if $|v|$ is sufficiently small, where $c_0 \neq 0$ is a constant.

Applying the Fourier inversion formula to $\nu_{\delta, l}^\lambda$ (cf. (6.9)), we thus find that

$$\begin{aligned} \nu_{\delta, l}^\lambda(x) &= \int |\xi_3|^{-\frac{1}{2}} \sigma_{-1/2}(\xi) \chi_1((2^{-l}\lambda)^{\frac{2}{3}} B_1) \chi_1\left(\frac{s_1 \xi_3}{\lambda}\right) \chi_1\left(\frac{s_2 \xi_3}{\lambda}\right) \chi_1\left(\frac{\xi_3}{\lambda}\right) e^{-i\xi_3 B_0(s, \delta, \sigma)} e^{i\xi \cdot x} \\ &\quad |B_1|^{-\frac{1}{4}} a_+ (|B_1|^{\frac{1}{2}}, s, \delta, \sigma) e^{-i|\xi_3| |B_1|^{\frac{3}{2}} q_+ (|B_1|^{\frac{1}{2}}, s, \delta, \sigma)} d\xi, \end{aligned}$$

where again $B_1 = B_1(s, \delta, \sigma)$. Changing as before coordinates from $\xi = (\xi_1, \xi_2, \xi_3)$ to s_1, s_2 and $s_3 := \xi_3/\lambda$, i.e.,

$$\xi_1 = s_1 \xi_3 = \lambda s_1 s_3, \quad \xi_2 = \lambda s_2 \xi_3 = \lambda s_2 s_3, \quad \xi_3 = \lambda s_3,$$

we obtain, with σ_λ and $\tilde{\chi}$ essentially as before, that

$$\begin{aligned} \nu_{\delta, l}^\lambda(x) &= \lambda^2 \int \sigma_\lambda(s) \chi_1((2^{-l}\lambda)^{\frac{2}{3}} B_1) e^{-i\lambda s_3 \left(B_0(s', \delta, \sigma) + |B_1|^{\frac{3}{2}} q_+ (|B_1|^{\frac{1}{2}}, s', \delta, \sigma) - s_1 x_1 - s_2 x_2 - x_3 \right)} \\ &\quad |B_1|^{-\frac{1}{4}} a_+ (|B_1|^{\frac{1}{2}}, s', \delta, \sigma) \tilde{\chi}(s) ds. \end{aligned}$$

Observe that we can apply the same arguments that we used in the previous subsection in order to show that when $|x| \gg 1$, then $|\nu_{\delta, l}^\lambda(x)| \leq C_N \lambda^{-N}$ for every $N \in \mathbb{N}$. Again, we may thus assume in the sequel that $|x| \lesssim 1$.

We perform yet another change of coordinates, from $s' = (s_1, s_2)$ to (z, s_2) , where now

$$z := (2^{-l}\lambda)^{\frac{2}{3}} B_1(s', \delta, \sigma).$$

Here, we find that (compare (6.16))

$$(6.26) \quad s_1 = s_2^{\frac{n-1}{n-2}} G_3(s_2, \delta, \sigma) - (2^{-l}\lambda)^{-\frac{2}{3}} z,$$

and in particular

$$(6.27) \quad B_0(s, \delta, \sigma) = -(2^{-l}\lambda)^{-\frac{2}{3}}z s_2^{\frac{1}{n-2}}G_1(s_2, \delta, \sigma) + s_2^{\frac{n}{n-2}}(G_1G_3 - G_2)(s_2, \delta, \sigma).$$

We may thus re-write

$$(6.28) \quad \begin{aligned} \nu_{\delta,l}^\lambda(x) &= 2^{\frac{l}{2}}\lambda^{\frac{3}{2}} \int e^{-i\lambda s_3 \Phi_l(z, s_2, x, \delta, \sigma)} a_+ \left((2^{-l}\lambda)^{-\frac{1}{3}}|z|^{\frac{1}{2}}, s, \delta, \sigma \right) \\ &\quad (\sigma_\lambda \tilde{\chi}) \left(s_2^{\frac{n-1}{n-2}}G_3(s_2, \delta, \sigma) - \lambda^{-\frac{2}{3}}z, s_2 \right) ds_2 |z|^{-\frac{1}{4}} \chi_1(z) dz, \end{aligned}$$

where here

$$(6.29) \quad \begin{aligned} \Phi_l(z, s_2, x, \delta, \sigma) &:= s_2^{\frac{n}{n-2}}(G_1G_3 - G_2)(s_2, \delta, \sigma) - s_2^{\frac{n-1}{n-2}}G_3(s_2, \delta, \sigma)x_1 - s_2x_2 - x_3 \\ &+ (2^{-l}\lambda)^{-\frac{2}{3}}z \left(x_1 - s_2^{\frac{1}{n-2}}G_1(s_2, \delta, \sigma) \right) \\ &+ (2^{-l}\lambda)^{-1}|z|^{\frac{3}{2}}q_+ \left((2^{-l}\lambda)^{-\frac{1}{3}}|z|^{\frac{1}{2}}, s_2^{\frac{n-1}{n-2}}G_3(s_2, \delta, \sigma) - (2^{-l}\lambda)^{-\frac{2}{3}}z, s_2, \delta, \sigma \right). \end{aligned}$$

Now, observe that

$$1 \ll M_1 \leq 2^{-l}\lambda, \quad 2^l = \lambda(2^{-l}\lambda)^{-1} \ll \lambda(2^{-l}\lambda)^{-\frac{2}{3}} = \lambda^{\frac{1}{3}}2^{\frac{2l}{3}},$$

if we choose M_1 sufficiently large, and that $|z| \sim 1, |s_2| \sim 1$ in the integral (6.29), whereas $|x_1| \lesssim 1$.

In order to control the integral in z , we therefore split the domain of integration for s_2 in (6.28) into the region where $\lambda(2^{-l}\lambda)^{-\frac{2}{3}}|x_1 - s_2^{\frac{1}{n-2}}G_1(s_2, \delta, \sigma)| \geq C2^l$, where $C \gg 1$ is a sufficiently large constant, and its complement. Notice that each of these regions consists of at most four intervals.

On the first s_2 -region, we integrate by parts in z twice. This allows to estimate the corresponding contribution to $\nu_{\delta,l}^\lambda(x)$ by a constant times

$$\begin{aligned} &\int_{\lambda^{\frac{1}{3}}2^{\frac{2l}{3}}|x_1 - s_2^{\frac{1}{n-2}}G_1(s_2, \delta, \sigma)| \geq C2^l, |s_2| \sim 1} \frac{2^{\frac{l}{2}}\lambda^{\frac{3}{2}}}{(\lambda^{\frac{1}{3}}2^{\frac{2l}{3}}|x_1 - s_2^{\frac{1}{n-2}}G_1(s_2, \delta, \sigma)|)^2} ds_2 \\ &\lesssim 2^{\frac{l}{2}}\lambda^{\frac{3}{2}} \int_{\lambda^{\frac{1}{3}}2^{\frac{2l}{3}}|v| \geq C2^l} \frac{dv}{(\lambda^{\frac{1}{3}}2^{\frac{2l}{3}}|v|)^2} \lesssim 2^{\frac{l}{2}}\lambda^{\frac{3}{2}}2^{-l}(\lambda^{\frac{1}{3}}2^{\frac{2l}{3}})^{-1} = 2^{-\frac{7l}{6}}\lambda^{\frac{7}{6}}, \end{aligned}$$

which is even better than needed.

On the complementary s_2 intervals, the coefficient of z in (6.29) is controlled by the coefficient of $|z|^{3/2}$. Thus, if there is no critical point with respect to z , we may integrate by parts once, and if there is a critical point, we may apply the method of stationary phase in z , by which we gain at least a factor $2^{-l/2}$, and subsequently apply the van der Corput type Lemma 2.2 to the s_2 -integration in the same way as we did in

the preceding subsection, in order to see that the corresponding contributions to $\nu_{\delta,l}^\lambda(x)$ can be estimated by

$$C 2^{\frac{l}{2}} \lambda^{\frac{3}{2}} 2^{-\frac{l}{2}} \lambda^{-\frac{1}{3}}.$$

In combination, these estimates yield (6.22), which concludes the proof of Lemma 6.5.

7. THE CASE WHEN $h_{\text{lin}}(\phi) \geq 2$: PREPARATORY RESULTS

Recall that $h = h(\phi) > 2$ when $h_{\text{lin}} \geq 2$, and that we assume that the original coordinates x are linearly adapted, so that $d = h_{\text{lin}} \geq 2$. Moreover, based on Varchenko's algorithm, we can locally find an adapted coordinate system $y_1 = x_1$, $y_2 = x_2 - \psi(x_1)$ for the function ϕ near the origin. In these coordinates, ϕ is given by $\phi^a(y) := \phi(y_1, y_2 + \psi(y_1))$ (cf. (1.8), (1.9)).

Also recall that the vertices of the Newton polyhedron $\mathcal{N}(\phi^a)$ of ϕ^a are assumed to be the points (A_l, B_l) , $l = 0, \dots, n$, so that the Newton polyhedron $\mathcal{N}(\phi^a)$ is the convex hull of the set $\bigcup_l ((A_l, B_l) + \mathbb{R}_+^2)$, where $A_{l-1} < A_l$ for every $l \geq 1$. Moreover, $L_l := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1^l t_1 + \kappa_2^l t_2 = 1\}$ denotes the line passing through the points (A_{l-1}, B_{l-1}) and (A_l, B_l) , and $a_l = \kappa_2^l / \kappa_1^l$. The a_l can be identified as the distinct leading exponents of all the roots of ϕ^a in case that ϕ^a is analytic (see Section 3 of [13]), and the cluster of roots whose leading exponent in their Puiseux series expansion is given by a_l is associated to the edge $\gamma_l = [(A_{l-1}, B_{l-1}), (A_l, B_l)]$ of $\mathcal{N}(\phi^a)$.

As before, following Subsection 8.2 of [13], we choose the integer $l_0 \geq 1$ such that

$$a_0 < \dots < a_{l_0-1} \leq m < a_{l_0} < \dots < a_l < a_{l+1} < \dots < a_n.$$

As has been shown in Section 3 of [13], the vertex (A_{l_0-1}, B_{l_0-1}) lies strictly above the bisectrix, i.e., $A_{l_0-1} < B_{l_0-1}$, since the original coordinates x were assumed to be non-adapted.

Following in a slightly modified way the discussion in Section 3 of [13] we single out a particular edge by fixing the corresponding index $l_{\text{pr}} \geq l_0$:

Cases:

- (a) If the principal face of ϕ^a is a compact edge, we choose l_{pr} so that the edge $\gamma_{l_{\text{pr}}} = [(A_{l_{\text{pr}}-1}, B_{l_{\text{pr}}-1}), (A_{l_{\text{pr}}}, B_{l_{\text{pr}}})]$ is the principal face $\pi(\phi^a)$ of the Newton polyhedron of ϕ^a .
- (b) If $\pi(\phi^a)$ is the vertex (h, h) , we choose l_{pr} so that $(h, h) = (A_{l_{\text{pr}}-1}, B_{l_{\text{pr}}-1})$. Then (h, h) is the right endpoint of the compact edge $\gamma_{l_{\text{pr}}-1}$.
- (c) If the principal face $\pi(\phi^a)$ is unbounded, i.e., a half-line given by $t_1 \geq A$ and $t_2 = h := B$, with $A < B$, then we distinguish two sub-cases:
 - (c1) If the point (A, B) is the right endpoint of a compact edge of $\mathcal{N}(\phi^a)$, then we choose again l_{pr} so that this edge is given by $\gamma_{l_{\text{pr}}-1}$.
 - (c2) Otherwise, (A, B) is the only vertex of $\mathcal{N}(\phi^a)$, i.e., $\mathcal{N}(\phi^a) = (A, B) + \mathbb{R}_+^2$.

We also put

$$(7.1) \quad a := \begin{cases} a_{l_{\text{pr}}} & \text{in Case (a);} \\ a_{l_{\text{pr}}-1} & \text{in Case (b) and Case (c1);} \\ m & \text{in Case (c2).} \end{cases}$$

Following [13] and [14], in the cases (a) - (c1) we shall decompose the domain (4.1) in which ρ_1 is supported into subdomains

$$D_l := \{(x_1, x_2) : \varepsilon_l x_1^{a_l} < |x_2 - \psi(x_1)| \leq N_l x_1^{a_l}\}, \quad l = l_0, \dots, l_{\text{pr}} - 1,$$

which correspond to the κ^l -homogeneous domains $D_l^a := \{(y_1, y_2) : \varepsilon_l y_1^{a_l} < |y_2| \leq N_l y_1^{a_l}\}$ in our adapted coordinates y , and intermediate “transition” domains

$$E_l := \{(x_1, x_2) : N_{l+1} x_1^{a_{l+1}} < |x_2 - \psi(x_1)| \leq \varepsilon_l x_1^{a_l}\},$$

where $l = l_0, \dots, l_{\text{pr}} - 1$ in Case (a), and $l = l_0, \dots, l_{\text{pr}} - 2$ in all other cases, as well as the “first” transition domain

$$E_{l_0-1} := \{(x_1, x_2) : N_{l_0} x_1^{a_{l_0}} < |x_2 - \psi(x_1)| \leq \varepsilon_{l_0} x_1^m\},$$

corresponding to the y -domains $E_l^a := \{(y_1, y_2) : N_{l+1} y_1^{a_{l+1}} < |y_2| \leq \varepsilon_l y_1^{a_l}\}$, respectively $E_{l_0-1}^a := \{(y_1, y_2) : N_{l_0} y_1^{a_{l_0}} < |y_2| \leq \varepsilon_{l_0} y_1^m\}$. Here, the $\varepsilon_l > 0$ are small and the $N_l > 0$ are large parameters to be determined later. We remark that the domain E_{l_0-1} can be written like E_l with $l = l_0 - 1$ if we replace, with some slight abuse of notation, a_{l_0-1} by m and κ_{l_0-1} by κ . We shall make use of this unified way of describing E_l in the sequel.

What will remain after removing these domains is a domain of the form

$$(7.2) \quad D_{\text{pr}} := \begin{cases} \{(x_1, x_2) : |x_2 - \psi(x_1)| \leq N x_1^a\} & \text{in Case (a);} \\ \{(x_1, x_2) : |x_2 - \psi(x_1)| \leq \varepsilon x_1^a\}, & \text{in all other cases,} \end{cases}$$

where N is sufficiently large and ε sufficiently small.

In the cases (c1) and (c2), we shall furthermore also regard the domains

$$(7.3) \quad E_{l_{\text{pr}}-1} := D_{\text{pr}} = \{(x_1, x_2) : |x_2 - \psi(x_1)| \leq \varepsilon x_1^a\}$$

as “generalized” transition domains. Notice that in the Case (c2) this domain will cover the domain in (4.1), since here $a = m$, so that the proof of Proposition 4.3 will be complete once we shall have handled all these transitions domains in the next section. In a similar way, the discussion of Case (c1) will be complete once we have handled the domains E_l and D_l . This will eventually reduce our problem to studying the domain D_{pr} in the cases (a) and (b).

8. RESTRICTION ESTIMATES IN THE TRANSITION DOMAINS WHEN $h_{\text{lin}}(\phi) \geq 2$

Following a standard approach, we would like to study the contributions of the domains E_l by means of a decomposition of the corresponding y -domains E_l^a into dyadic rectangles. These rectangles correspond to a kind of “curved boxes” in the

original coordinates x , so that we cannot achieve the localization to them by means of Littlewood-Paley decompositions in the variables x_1 and x_2 . However, the following lemma shows that this localization can nevertheless be induced by means of Littlewood-Paley decompositions in the variables x_1 and x_3 .

We shall formulate this lemma for a general smooth, finite type function Φ with $\Phi(0,0) = 0$ and $\nabla\Phi(0,0) = 0$ in place of ϕ^a , since it will be applied not only to ϕ^a . However, we shall keep the notation introduced for ϕ^a , denoting for instance by (A_l, B_l) , $l = 0, \dots, n$ the vertices of the Newton polyhedron of Φ , by κ^l the weight associated to the edge $\gamma_l = [(A_{l-1}, B_{l-1}), (A_l, B_l)]$, etc..

Lemma 8.1. *For $l \geq l_0$, let $[(A_{l-1}, B_{l-1}), (A_l, B_l)]$ and $[(A_l, B_l), (A_{l+1}, B_{l+1})]$ be two subsequent compact edges of $\mathcal{N}(\Phi)$, with common vertex (A_l, B_l) , and associated weights κ^l and κ^{l+1} . Recall also that $a_l = \kappa_2^l / \kappa_1^l < a_{l+1} = \kappa_2^{l+1} / \kappa_1^{l+1}$. For a given $M > 0$, and $\delta > 0$ sufficiently small, consider the domain*

$$E^a := \{(y_1, y_2) : 0 < y_1 < \delta, 2^M y_1^{a_{l+1}} < |y_2| \leq 2^{-M} y_1^{a_l}\}.$$

(a) *There is a constant $C > 0$ such that*

$$(8.1) \quad \Phi(y) = c_{A_l, B_l} y_1^{A_l} y_2^{B_l} \left(1 + O(\delta^C + 2^{-M})\right) \quad \text{on } E^a,$$

where c_{A_l, B_l} denotes the Taylor coefficient of Φ corresponding to (A_l, B_l) . More precisely, $\Phi(y) = c_{A_l, B_l} y_1^{A_l} y_2^{B_l} (1 + g(y))$, where $|g^{(\beta)}(y)| \leq C_\beta (\delta^C + 2^{-M}) |y_1^{-\beta_1} y_2^{-\beta_2}|$ for every multi-index $\beta \in \mathbb{N}^2$.

(b) *For $M, j \in \mathbb{N}$ sufficiently large, the following conditions are equivalent:*

- (i) $y_1 \sim 2^{-j}$, $(y_1, y_2) \in E^a$ and $2^{A_l j + B_l k} \Phi(y) \sim 1$;
- (ii) $y_1 \sim 2^{-j}$, $y_2 \sim 2^{-k}$ and $a_l j + M \leq k \leq a_{l+1} j - M$.

Moreover, if we set $\phi_{j,k}(x) := 2^{A_l j + B_l k} \Phi(2^{-j} x_1, 2^{-k} x_2)$, then under the previous conditions we have that $\phi_{j,k}(x) = c_{A_l, B_l} x_1^{A_l} x_2^{B_l} \left(1 + O(2^{-Cj} + 2^{-M})\right)$ on the set where $x_1 \sim 1, |x_2| \sim 1$, in the sense of the C^∞ -topology.

The statements in (a) and (b) remains valid also in the case where $l = l_0 - 1$.

Proof. When Φ is analytic, these results have essentially been proven in Section 8.3 of [13], at least implicitly. We shall here give an elementary proof which works also for smooth functions Φ .

We begin with the case where $l > l_0$. Notice first that (b) is an immediate consequence of (a). In order to prove (a), let us denote by Φ_N the Taylor polynomial of degree N of Φ centered at the origin. Since $(\Phi - \Phi_N)(y_1, y_2) = O(|y_1|^N + |y_2|^N)$, it is easily seen that $y_1^{-A_l} y_2^{-B_l} (\Phi - \Phi_N)(y_1, y_2) = O(2^{-B_l M})$ on E^a , provided N is sufficiently large and δ small. It therefore suffices to prove (8.1) for Φ_N in place of Φ .

If $\Phi((y_1, y_2) \sim \sum_{\alpha_1, \alpha_2=0}^{\infty} c_{\alpha_1, \alpha_2} y_1^{\alpha_1} y_2^{\alpha_2}$ is the Taylor series of Φ centered at the origin, then we decompose the polynomial Φ_N as $\Phi_N = P^+ + P^-$, where

$$P^+(y_1, y_2) := \sum_{\alpha_1 + \alpha_2 \leq N, \alpha_2 > B_l} c_{\alpha_1, \alpha_2} y_1^{\alpha_1} y_2^{\alpha_2}, \quad P^-(y_1, y_2) := \sum_{\alpha_1 + \alpha_2 \leq N, \alpha_2 \leq B_l} c_{\alpha_1, \alpha_2} y_1^{\alpha_1} y_2^{\alpha_2}.$$

Let (α_1, α_2) be one of the multi-indices appearing in P^- , and assume it is different from (A_l, B_l) . Let $(y_1, y_2) \in E^a$, and assume, for notational convenience, that $y_2 > 0$. Since clearly $A_l, B_l > 0$, we have

$$\frac{y_1^{\alpha_1} y_2^{\alpha_2}}{y_1^{A_l} y_2^{B_l}} = y_1^{\alpha_1 - A_l} y_2^{\alpha_2 - B_l} \leq y_1^{\alpha_1 - A_l} \left(2^M y_1^{a_{l+1}} \right)^{\alpha_2 - B_l} = 2^{(\alpha_2 - B_l)M} y_1^{\alpha_1 + a_{l+1}\alpha_2 - (A_l + a_{l+1}B_l)}.$$

It is easy to see that $A_l + a_{l+1}B_l = A_{l+1} + a_{l+1}B_{l+1}$, so that

$$(8.2) \quad \frac{y_1^{\alpha_1} y_2^{\alpha_2}}{y_1^{A_l} y_2^{B_l}} \leq 2^{(\alpha_2 - B_l)M} y_1^{\alpha_1 + a_{l+1}\alpha_2 - (A_{l+1} + a_{l+1}B_{l+1})}.$$

But, since γ_{l+1} is an edge of $\mathcal{N}(\Phi)$, we have that $\kappa_1^{l+1}\alpha_1 + \kappa_2^{l+1}\alpha_2 \geq 1$, i.e., $\alpha_1 + a_{l+1}\alpha_2 \geq (\kappa_1^{l+1})^{-1}$, whereas $A_{l+1} + a_{l+1}B_{l+1} = (\kappa_1^{l+1})^{-1}$. Thus, (8.2) implies that $y_1^{\alpha_1} y_2^{\alpha_2} \leq 2^{(\alpha_2 - B_l)M} y_1^{A_l} y_2^{B_l}$, so that $y_1^{\alpha_1} y_2^{\alpha_2} \leq 2^{-M} y_1^{A_l} y_2^{B_l}$ when $\alpha_2 < B_l$. And, when $\alpha_2 = B_l$, then (α_1, α_2) lies in the interior of $\mathcal{N}(\Phi)$, so that $\alpha_1 + a_{l+1}\alpha_2 - (A_{l+1} + a_{l+1}B_{l+1}) > 0$, hence $y_1^{\alpha_1} y_2^{\alpha_2} \leq \delta^C y_1^{A_l} y_2^{B_l}$ for some positive constant C .

The estimates of the derivatives of $g(y) = \Phi(y)/c_{A_l, B_l} y_1^{A_l} y_2^{B_l} - 1$ follow in a very similar way.

The terms in P^+ can be estimated analogously, making use here of the estimates $y_2 \leq 2^{-M} y_1^{a_l}$ and $\kappa_1^l \alpha_1 + \kappa_2^l \alpha_2 \geq 1$. This proves (a).

Finally, if $l = l_0$, exactly the same arguments work, if we re-define a_{l_0-1} to be m and κ_{l_0-1} to be κ , since $\kappa_2/\kappa_1 = m$. Q.E.D.

A similar result applies also to the generalized transition domains $E_{l_{\text{pr}}-1}$ arising in the cases (c1) and (c2), provided we can factor the root $y_2 = 0$ to its given order, which applies in particular when Φ is real-analytic (some easy examples show that it may be false otherwise). Recall that in these cases, the principal face of $\mathcal{N}(\phi^a)$ is an unbounded half-line with left endpoint (A, B) . More generally, we have the following result:

Lemma 8.2. *Assume that (A, B) is a vertex of $\mathcal{N}(\Phi)$ such that the unbounded horizontal half-line with left endpoint (A, B) is a face of $\mathcal{N}(\Phi)$, and assume in addition that Φ factors as $\Phi(y_1, y_2) = y_2^B \Upsilon(y_1, y_2)$, with a smooth function Υ . Moreover, let $L_\kappa := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1 t_1 + \kappa_2 t_2 = 1\}$ be a non-horizontal support line for $\mathcal{N}(\Phi)$ (i.e., $\kappa_1 > 0$) passing through (A, B) , and let $a := \kappa_1/\kappa_1$. We then put*

$$E^a := \{(y_1, y_2) : 0 < y_1 < \delta, |y_2| \leq 2^{-M} y_1^a\}.$$

(a) *There is a constant $C > 0$ such that*

$$(8.3) \quad \Phi(y) = c_{A,B} y_1^A y_2^B \left(1 + O(\delta^C + 2^{-M}) \right) \quad \text{on } E^a,$$

where $c_{A,B}$ denotes the Taylor coefficient of Φ corresponding to (A, B) . More precisely, $\Phi(y) = c_{A,B} y_1^A y_2^B (1 + g(y))$, where $|g^{(\beta)}(y)| \leq C_\beta (\delta^C + 2^{-M}) |y_1^{-\beta_1} y_2^{-\beta_2}|$ for every multi-index $\beta \in \mathbb{N}^2$.

(b) *For $M, j \in \mathbb{N}$ sufficiently large, the following conditions are equivalent:*

- (i) $y_1 \sim 2^{-j}$, $(y_1, y_2) \in E^a$ and $2^{Aj+Bk}\Phi(y) \sim 1$;
- (ii) $y_1 \sim 2^{-j}$, $y_2 \sim 2^{-k}$ and $aj + M \leq k$.

Moreover, if we set $\phi_{j,k}(x) := 2^{Aj+Bk}\Phi(2^{-j}x_1, 2^{-k}x_2)$, then under the previous conditions we have that $\phi_{j,k}(x) = c_{A,B}x_1^A x_2^B (1 + O(2^{-Cj} + 2^{-M}))$ on the set where $x_1 \sim 1, |x_2| \sim 1$, in the sense of the C^∞ - topology.

Proof. It suffices again to prove (a).

By our assumption, $\Phi(y_1, y_2) = y_2^B \Upsilon(y_1, y_2)$, so that $\Phi(y)/y_1^A y_2^B = \Upsilon(y)/y_1^A$. Approximating Υ by its Taylor polynomial of sufficiently high degree, we again see that we may reduce to the case where Υ , hence Φ , is a polynomial. Then let (α_1, α_2) be any point different from (A, B) in its Taylor support. Since $\alpha_2 \geq B$, assuming again that $y_2 > 0$, we see that

$$\frac{y_1^{\alpha_1} y_2^{\alpha_2}}{y_1^A y_2^B} = y_1^{\alpha_1-A} y_2^{\alpha_2-B} \leq y_1^{\alpha_1-A} \left(2^{-M} y_1^a\right)^{\alpha_2-B} = 2^{-(\alpha_2-B)M} y_1^{\alpha_1+a\alpha_2-(A+aB)}.$$

Moreover, clearly $\alpha_1 + a\alpha_2 \geq A + aB$, and $\alpha_1 + a\alpha_2 > A + aB$ when $\alpha_2 = B$. We can thus argue in a very similar way as in the proof of Lemma 8.1 to finish the proof.

Q.E.D.

Let us now fix $l \in \{l_0 - 1, \dots, l_{\text{pr}} - 1\}$, and consider the corresponding (generalized) transition domain E_l from Section 7, which can be written as

$$E_l = \{(x_1, x_2) : Nx_1^{a_{l+1}} < |x_2 - \psi(x_1)| \leq \varepsilon x_1^{a_l}\},$$

where, with some slight abuse of notation, we have again re-defined $a_{l_0-1} := m$, and put $a_{l_{\text{pr}}} := \infty$ in the cases (c1) and (c2), so that $x_1^{a_{l_{\text{pr}}}} := 0$, by definition.

Following [13], we shall localize to the domain E_l by means of a cut-off function

$$\tau_l(x_1, x_2) := \chi_0\left(\frac{x_2 - \psi(x_1)}{\varepsilon x_1^{a_l}}\right) (1 - \chi_0)\left(\frac{x_2 - \psi(x_1)}{Nx_1^{a_{l+1}}}\right),$$

where $\chi_0 \in C_0^\infty(\mathbb{R})$ is again supported in $[-1, 1]$ and $\chi_0 \equiv 1$ on $[-1/2, 1/2]$ (actually, χ_0 may depend on l). In Case (c), when $l = l_{\text{pr}} - 1$ and $a_{l_{\text{pr}}} = \infty$, the second factor has to be interpreted as 1, i.e.,

$$\tau_{l_{\text{pr}}-1}(x_1, x_2) = \chi_0\left(\frac{x_2 - \psi(x_1)}{\varepsilon x_1^{a_l}}\right).$$

Recall that ϕ is assumed to satisfy Condition (R).

Proposition 8.3. *Let $l \in \{l_0 - 1, \dots, l_{\text{pr}} - 1\}$. Then, if $\varepsilon > 0$ is chosen sufficiently small and $N > 0$ sufficiently large,*

$$\left(\int_S |\widehat{f}|^2 d\mu^{\tau_l}\right)^{1/2} \leq C_p \|f\|_{L^p(\mathbb{R}^3)}, \quad f \in \mathcal{S}(\mathbb{R}^3),$$

whenever $p' \geq p'_c$.

Proof. Consider partitions of unity $\sum_j \chi_j(s) = 1$ and $\sum_k \tilde{\chi}_{j,k}(s) = 1$ on $\mathbb{R} \setminus \{0\}$ with $\chi, \tilde{\chi} \in C_0^\infty(\mathbb{R})$ supported in $[-2, -1/2] \cup [1/2, 2]$ respectively $[-2^{B_l}, -2^{-B_l}] \cup [2^{-B_l}, 2^{B_l}]$, where $\chi_j(s) := \chi(2^j s)$ and, for j fixed, $\tilde{\chi}_{j,k}(s) := \chi(2^{A_l j + B_l k} s)$, and let

$$\chi_{j,k}(x_1, x_2, x_3) := \chi_j(x_1) \tilde{\chi}_{j,k}(x_3) = \chi(2^j x_1) \tilde{\chi}(2^{A_l j + B_l k} x_3), \quad j, k \in \mathbb{Z}.$$

Notice here that $B_l > B_{l+1} \geq 0$. We next put $\mu_{j,k} := \chi_{j,k} \mu^{\tau_l}$, and assume that μ has sufficiently small support near the origin. Then clearly $\mu_{j,k} = 0$, unless $j \geq j_0$, where $j_0 > 0$ is a large number which we can still choose suitably later. But then, according to Lemma 8.1, we may assume in addition that

$$(8.4) \quad a_l j + M \leq k \leq a_{l+1} j - M,$$

where M is a large number. Indeed, we may choose $N := 2^M$ and $\varepsilon := 2^{-M}$, and then Lemma 8.1 (b) shows that $\mu_{j,k} = 0$ for all pairs (j, k) not satisfying (8.4). Notice that this also implies that $k \geq k_0$ for some large number k_0 . Observe also that the measure $\mu_{j,k}$ is supported over a “curved box” given by $x_1 \sim 2^{-j}$ and $|x_2 - \psi(x_1)| \lesssim 2^{-k}$. This shows that the localization that we have achieved by means of the cut-off function $\chi_{j,k}$ is very similar to the localization that we could have imposed by means of the cut-off function $\chi(2^{-j} x_1) \chi(2^{-k}(x_2 - \psi(x_1)))$.

Then, applying again Littlewood-Paley theory, now in the variables x_1 and x_3 , and interpolating with the trivial $L^1 \rightarrow L^\infty$ estimate for the Fourier transform, we see that in order to prove Proposition 8.3, it suffices to prove uniform restriction estimates for the measures $\mu_{j,k}$ at the critical exponent, i.e., that

$$(8.5) \quad \int_S |\widehat{f}|^2 d\mu_{j,k} \leq C \|f\|_{L^{p_c}(\mathbb{R}^3)}^2, \quad \text{when } (j, k) \text{ satisfies (8.4) and } j \geq j_0,$$

provided M and j_0 are chosen sufficiently large.

We introduce the normalized measures $\nu_{j,k}$ given by

$$\langle \nu_{j,k}, f \rangle := \int f(x_1, 2^{mj-k} x_2 + x_1^m \omega(2^{-j} x_1), \phi_{j,k}(x_1, x_2)) a_{j,k}(x) dx,$$

where

$$\begin{aligned} a_{j,k}(x) = \eta\left(2^{-j} x_1, 2^{-k} x_2 + \psi(2^{-j} x_1)\right) & \left[\chi_0\left(2^{a_l j + M - k} \frac{x_2}{x_1^{a_l}}\right) (1 - \chi_0)\left(2^{a_{l+1} j - M - k} \frac{x_2}{x_1^{a_{l+1}}}\right) \right] \\ & \times \chi(x_1) \tilde{\chi}(\phi_{j,k}(x_1, x_2)). \end{aligned}$$

Here, according to Lemma 8.1, the functions $\phi_{j,k}$ satisfy

$$\phi_{j,k}(x_1, x_2) = c x_1^{A_l} x_2^{B_l} + O(2^{-M}) \quad \text{in } C^\infty$$

on domains where $x_1 \sim 1, |x_2| \sim 1$, and the amplitude $a_{j,k}$ in the integral above is supported in such a domain.

Observe that

$$(8.6) \quad \langle \mu_{j,k}, f \rangle = 2^{-j-k} \int f(2^{-j}y_1, 2^{-mj}y_2, 2^{-(A_lj+B_lk)}y_3) d\nu_{j,k}(y),$$

which follows easily by means of a change to adapted coordinates in the integral defining the measure $\mu_{j,k}$ and scaling in x_1 by the factor 2^{-j} and in x_2 by the factor 2^{-k} .

We observe that the measure $\nu_{j,k}$ is supported on the surface given by

$$S_{j,k} := \{(x_1, 2^{mj-k}x_2 + x_1^m\omega(2^{-j}x_1), \phi_{j,k}(x_1, x_2)) : x_1 \sim 1 \sim x_2\}.$$

which is a small perturbation of the limiting surface

$$S_\infty := \{(x_1, x_1^m\omega(0), cx_1^{A_l}x_2^{B_l}) : x_1 \sim 1 \sim x_2\},$$

since $mj - k \leq a_lj - k \leq -M$ because of (8.4). Notice also that $|\partial(cx_1^{A_l}x_2^{B_l})/\partial x_2| \sim 1$, since $B_l \geq 1$. This show that S_∞ and hence also $S_{j,k}$ (for j and M sufficiently large) is a smooth hypersurface with one non-vanishing principal curvature (with respect to x_1) of size ~ 1 . This implies that

$$|\widehat{\nu_{j,k}}(\xi)| \leq C(1 + |\xi|)^{-1/2},$$

uniformly in j and k .

Moreover, the total variations of the measures $\nu_{j,k}$ are uniformly bounded, i.e., $\sup_{j,k} \|\nu_{j,k}\|_1 < \infty$.

We may thus apply again Greenleaf's result [10] in order to prove that

$$(8.7) \quad \int |\widehat{f}|^2 d\nu_{j,k} \leq C \|f\|_{L^p(\mathbb{R}^3)}^2$$

holds, whenever $p' \geq 6$, with a constant C which is independent of j, k . Since $p'_c \geq 2d + 2 \geq 6$, this holds in particular for $p = p_c$. Re-scaling this estimate by means of (8.6), this implies

$$(8.8) \quad \int |\widehat{f}|^2 d\mu_{j,k} \leq C 2^{-j-k+2\frac{(m+1+A_l)j+B_lk}{p'_c}} \|f\|_{L^{p_c}(\mathbb{R}^3)}^2.$$

But, we can write k in the form $k = \theta a_lj + (1 - \theta)a_{l+1}j + \tilde{M}$ with $0 \leq \theta \leq 1$ and $|\tilde{M}| \leq M$. Then

$$\begin{aligned} -j - k + 2\frac{(m+1+A_l)j+B_lk}{p'_c} &= -j\theta \left[1 + a_l - 2\frac{m+1+A_l+a_lB_l}{p'_c} \right] \\ &\quad -j(1-\theta) \left[1 + a_{l+1} - 2\frac{m+1+A_l+a_{l+1}B_l}{p'_c} \right] + \left(-1 + 2\frac{B_l}{p'_c} \right) \tilde{M}. \end{aligned}$$

Recall next that by the definition of the r -height and the critical exponent p'_c , we have $p'_c \geq 2(h_l + 1)$ whenever $l \geq l_0$. And, (1.11) shows that

$$(8.9) \quad h_l + 1 = \frac{1 + (1+m)\kappa_1^l}{|\kappa^l|} = \frac{m+1 + \frac{1}{\kappa_1^l}}{1 + a_l}.$$

Moreover, we have seen in the proof of Lemma (8.1) that $A_l + a_l B_l = 1/\kappa_1^l$, so that

$$2(h_l + 1) = 2 \frac{m + 1 + A_l + a_l B_l}{1 + a_l}.$$

We thus find that $1 + a_l - 2(m + 1 + A_l + a_l B_l)/p'_c \geq 0$. Arguing in a similar way for $l + 1$ in place of l , by using that $p'_c \geq 2(h_{l+1} + 1)$ and $A_l + a_{l+1} B_l = 1/\kappa_1^{l+1}$ we also see that $1 + a_{l+1} - 2(m + 1 + A_l + a_{l+1} B_l)/p'_c \geq 0$.

Consequently, the exponent on the right-hand side of the estimate (8.8) is uniformly bounded from above, which verifies the claimed estimate (8.5).

Assume next that $l = l_0 - 1$. Observe that in this case, by following Varchenko's algorithm one observes that the left endpoint (A_{l_0-1}, B_{l_0-1}) of the edge $[(A_{l_0-1}, B_{l_0-1}), (A_{l_0}, B_{l_0})]$ of the Newton polyhedron of ϕ^a belongs also to the Newton polyhedron of ϕ and lies on the principal line $L = L_\kappa$ of $\mathcal{N}(\phi)$, whose slope is the reciprocal of $\kappa_2/\kappa_1 = m$. Thus, if we formally replace h_{l_0-1} by d in the previous argument (compare also Remark 1.3 (a)), it is easily seen that the previous argument works in exactly the same way.

What remains to be considered are the generalized transition domains $E_{l_{\text{pr}}-1}$ in the cases (c1) and (c2). Observe that in this case Condition (R) implies that $\Phi := \phi^a$ satisfies the factorization hypothesis of Lemma 8.2. We may therefore argue in a similar way as before, by applying Lemma 8.2 in place of Lemma 8.1, and obtain the estimate

$$(8.10) \quad \int_S |\widehat{f}|^2 d\mu_{j,k} \leq C 2^{-j-k+2 \frac{(m+1+A)j+Bk}{p'_c}} \|f\|_{L^{p_c}(\mathbb{R}^3)}^2,$$

where here $B = h$ is the height of ϕ , and where now we may only assume that

$$(8.11) \quad a_l j + M \leq k$$

Since, by the definition of the r -height, we have $p'_c \geq 2h_{l_{\text{pr}}-1} + 2 = 2B$ (compare (1.11)), we see that $-1 + \frac{2B}{p'_c} \leq 0$. We may thus estimate the exponent in (8.10) by

$$\begin{aligned} -j - k + 2 \frac{(m+1+A)j+Bk}{p'_c} &\leq -j \left[a + 1 - 2 \frac{m+1+A+aB}{p'_c} \right] + \left(-1 + \frac{2B}{p'_c} \right) M \\ &\leq -j \frac{a+1}{p'_c} \left[p'_c - 2 \frac{m+1+A+aB}{a+1} \right]. \end{aligned}$$

And, in the case (c1), arguing as before we see that $2(m+1+A+aB)/(a+1) = 2(h_{l_{\text{pr}}} + 1) \leq p'_c$.

Finally, in the case (c2), we have $m = a$. Moreover, the point (A, B) lies on the principal line L of $\mathcal{N}(\phi)$, so that $\kappa_1 A + \kappa_2 B = 1$, i.e., $A + aB = 1/\kappa_1$. This shows that

$$2 \frac{m+1+A+aB}{a+1} = 2 \left(1 + \frac{1}{\kappa_1 + \kappa_2} \right) = 2(1+d) \leq p'_c.$$

We thus see that the uniform estimate (8.5) is valid also for the generalized transition domains. Q.E.D.

9. RESTRICTION ESTIMATES IN THE DOMAINS D_l , $l < l_{\text{pr}}$, WHEN $h_{\text{lin}}(\phi) \geq 2$

We shall now consider the domains D_l , $l = l_0, \dots, l_{\text{pr}} - 1$, from Section 7, which are homogeneous in the adapted coordinates. Following again [13] we can localize to these domains by means of cut-off functions

$$\rho_l(x_1, x_2) := \chi_0\left(\frac{x_2 - \psi(x_1)}{Nx_1^{a_l}}\right) - \chi_0\left(\frac{x_2 - \psi(x_1)}{\varepsilon x_1^{a_l}}\right), \quad l = l_0, \dots, l_{\text{pr}} - 1,$$

where χ_0 is as in the previous section. Recall that such domains do appear only in the cases (a), (b) and (c1).

Proposition 9.1. *Let $h_{\text{lin}}(\phi) > 5$, and assume that $l < l_{\text{pr}}$. Then, if $\varepsilon > 0$ is chosen sufficiently small and $N > 0$ sufficiently large,*

$$\left(\int_S |\widehat{f}|^2 d\mu^{\rho_l}\right)^{1/2} \leq C_p \|f\|_{L^p(\mathbb{R}^3)}, \quad f \in \mathcal{S}(\mathbb{R}^3),$$

whenever $p' \geq p'_c$.

Proof. Similarly to the proof of Proposition 4.1, we denote by $\{\delta_r\}_{r>0}$ the dilations associated to the weight κ^l , i.e., $\delta_r y := (r^{\kappa_1^l} y_1, r^{\kappa_2^l} y_2)$, where by y we again denote our adapted coordinates. Recall that the κ^l -principal part $\phi_{\kappa^l}^a$ of ϕ^a is homogeneous of degree one with respect to these dilations, and that we are interested in a κ^l -homogeneous domain of the form $D_l^a = \{(y_1, y_2) : 0 < y_1 < \delta, \varepsilon y_1^{a_l} < |y_2| \leq Nx_1^{a_l}\}$ with respect to the y -coordinates, where $\delta > 0$ can still be chosen as small as we please.

We shall prove that, given any real number c_0 with $\varepsilon \leq c_0 \leq N$, there is some $\varepsilon' > 0$ such that the desired restriction estimate holds true on the domain $D(c_0)$ in x -coordinates corresponding to the homogeneous domain

$$D^a(c_0) := \{(y_1, y_2) : 0 < y_1 < \delta, |y_2 - c_0 y_1^{a_l}| \leq \varepsilon' y_1^{a_l}\}$$

in y -coordinates. Since we can cover the closure of D_l^a by a finite number of such narrow domains, this will imply Proposition 9.1.

We can localize to a domain like $D(c_0)$ by means of a cut-off function

$$\rho_{(c_0)}(x_1, x_2) := \chi_0\left(\frac{x_2 - \psi(x_1) - c_0 x_1^{a_l}}{\varepsilon' x_1^{a_l}}\right).$$

Let us again fix a suitable smooth cut-off function $\chi \geq 0$ on \mathbb{R}^2 supported in an annulus $\mathcal{A} := \{x \in \mathbb{R}^2 : 1/2 \leq |y| \leq R\}$ such that the functions $\chi_k^a := \chi \circ \delta_{2^k}$ form a partition of unity. In the original coordinates x , these correspond to the functions $\chi_k(x) := \chi_k^a(x_1, x_2 - \psi(x_1))$. We then decompose the measure $\mu^{\rho_{(c_0)}}$ dyadically as

$$(9.1) \quad \mu^{\rho_{(c_0)}} = \sum_{k \geq k_0} \mu_k,$$

where $\mu_k := \mu^{\chi_k \rho_{(c_0)}}$. Notice that by choosing the support of η sufficiently small, we can choose $k_0 \in \mathbb{N}$ as large as we need. It is also important to observe that this

decomposition can essentially be achieved by means of a dyadic decomposition with respect to the variable x_1 , which again allows to apply Littlewood-Paley theory!

Moreover, changing to adapted coordinates in the integral defining μ_k and scaling by $\delta_{2^{-k}}$ we find that

$$\begin{aligned} \langle \mu_k, f \rangle &= 2^{-k|\kappa^l|} \int f(2^{-\kappa_1^l k} x_1, 2^{-\kappa_2^l k} x_2 + 2^{-m\kappa_1^l k} x_1^m \omega(2^{-\kappa_1^l k} x_1), 2^{-k} \phi_k(x)) \\ &\quad \eta(\delta_{2^{-k}} x) \chi(x) \chi_0\left(\frac{x_2 - c_0 x_1^{a_l}}{\varepsilon' x_1^{a_l}}\right) dx, \end{aligned}$$

where

$$(9.2) \quad \phi_k(x) := 2^k \phi^a(\delta_{2^{-k}} x) = \phi_{\kappa^l}^a(x) + \text{error terms of order } O(2^{-\delta k})$$

with respect to the C^∞ topology (and $\delta > 0$).

We consider the corresponding normalized measure ν_k given by

$$\langle \nu_k, f \rangle := \int f(x_1, 2^{(m\kappa_1^l - \kappa_2^l)k} x_2 + x_1^m \omega(2^{-\kappa_1^l k} x_1), \phi_k(x)) \tilde{\eta}(x) dx,$$

with amplitude $\tilde{\eta}(x) := \eta(\delta_{2^{-k}} x) \chi(x) \chi_0\left((x_2 - c_0 x_1^{a_l})/(\varepsilon' x_1^{a_l})\right)$.

Observe that the support of the integrand is contained in the thin neighborhood

$$U(v) := \mathcal{A} \cap \{(x_1, x_2) : |x_2 - c_0 x_1^{a_l}| \leq 2\varepsilon' x_1^{a_l}\}$$

of $v = v(c_0) := (1, c_0)$, and that the measure ν_k is supported on the hypersurface

$$S_k := \{g_k(x_1, x_2) := (x_1, 2^{(m\kappa_1^l - \kappa_2^l)k} x_2 + x_1^m \omega(2^{-\kappa_1^l k} x_1), \phi_k(x_1, x_2)) : (x_1, x_2) \in U(v)\},$$

which, for k sufficiently large, is a small perturbation of the limiting variety

$$S_\infty := \{g_\infty(x_1, x_2) := (x_1, \omega(0)x_1^m, \phi_{\kappa^l}^a(x)) : (x_1, x_2) \in U(v)\},$$

since $m\kappa_1^l - \kappa_2^l < a_l \kappa_1^l - \kappa_2^l = 0$ and since ϕ^k tends to $\phi_{\kappa^l}^a$ because of (9.2). The corresponding limiting measure will be denoted by ν_∞ .

By Littlewood-Paley theory (applied to the variable x_1) and interpolation, in order to prove the desired restriction estimates for the measure $\mu^{\rho(c_0)}$, it suffices again to prove uniform restriction estimates for the measures μ_k , i.e.,

$$(9.3) \quad \left(\int |\widehat{f}|^2 d\mu_k \right)^{1/2} \leq C \|f\|_{L^{p_c}}.$$

with a constant C not depending on $k \geq k_0$. We shall obtain these by first proving restriction estimates for the measures ν_k .

Indeed, we shall prove that for ε' sufficiently small, the estimate

$$(9.4) \quad \left(\int |\widehat{f}|^2 d\nu_k \right)^{1/2} \leq C \|f\|_{L^{p_c}}$$

holds true, with a constant C which does not depend on k . Then, after re-scaling, estimate (9.4) implies the following estimate for μ_k :

$$(9.5) \quad \left(\int |\widehat{f}|^2 d\mu_k \right)^{1/2} \leq C 2^{-k \left(\frac{|\kappa^l|}{2} - \frac{\kappa_1^l(1+m)+1}{p'_c} \right)} \|f\|_{L^{p_c}}.$$

But, by (1.11) (resp. (8.9)) we have that

$$\frac{|\kappa^l|}{2} - \frac{\kappa_1^l(1+m)+1}{p'_c} = \frac{|\kappa^l|}{2} \left(1 - \frac{2(h_l+1)}{p'_c} \right),$$

where, by definition, $p'_c \geq 2(h_l+1)$. This shows that the exponent on the right-hand side of (9.5) is less or equal to zero, which verifies (9.3).

We turn to the proof of (9.4). Recall that $v = (1, c_0)$. Depending on the behavior of $\phi_{\kappa^l}^a$ near v , we shall distinguish between three cases.

1. Case. $\partial_2 \phi_{\kappa^l}^a(v) \neq 0$. This assumption implies that we may use $y_2 := \phi_{\kappa^l}^a(x_1, x_2)$ in place of x_2 as a new coordinate for S_∞ (which thus is a hypersurface, too), and then also for S_k , in place of x_2 , provided ε' is chosen small enough and k sufficiently large. Since $x_1 \sim 1$ on $U(v)$, this then shows that S_k is a hypersurface with one non-vanishing principal curvature. Therefore we can again apply Greenleaf's restriction theorem from [10] and obtain that for $p' \geq 6$ and k sufficiently large the estimate

$$\left(\int |\widehat{f}|^2 d\nu_k \right)^{1/2} \leq C_p \|f\|_{L^p}$$

holds true, with a constant C_p which does not depend on k . This applies in particular to $p = p_c$, which gives (9.4).

2. Case. $\partial_2 \phi_{\kappa^l}^a(v) = 0$. Then $v = (1, c_0)$ is a real root of $\partial_2 \phi_{\kappa^l}^a$, of multiplicity $B-1 \geq 1$, so that a Taylor expansion with respect to x_2 around c_0 and homogeneity show that

$$\partial_2 \phi_{\kappa^l}^a(x_1, x_2) = (x_2 - c_0 x_1^{a_l})^{B-1} \tilde{Q}(x_1, x_2),$$

where \tilde{Q} is a κ^l -homogenous smooth function in $U(v)$ such that $\tilde{Q}(v) \neq 0$. Integrating in x_2 , and making again use of the κ^l -homogeneity of $\phi_{\kappa^l}^a$, we find that

$$(9.6) \quad \phi_{\kappa^l}^a(x_1, x_2) = (x_2 - c_0 x_1^{a_l})^B x_2^{B_l} Q(x_1, x_2) + c_1 x_1^{1/\kappa_1^l},$$

where Q is a κ^l -homogenous smooth function such that $Q(1, c_0) \neq 0$ and $Q(1, 0) \neq 0$ (recall that $c_0 \neq 0$). Here, $c_1 \in \mathbb{R}$ could possibly be zero (iff $\nabla \phi_{\kappa^l}^a(v) = 0$).

We claim that

$$(9.7) \quad B < d/2,$$

where again $d = d(\phi)$. Indeed, observe first that the vertex (A_l, B_l) lies above or on the bi-sectrix, so that $1 = \kappa_1^l A_l + \kappa_2^l B_l \leq (\kappa_1^l + \kappa_2^l) B_l = B_l/d_l$, where $d_l := d_h(\phi_{\kappa^l}^a)$ denotes the homogenous distance of $\phi_{\kappa^l}^a$. But, since $a_l > m$, so that the edge γ_l is less steep than the line L (which intersects the bi-sectrix at (d, d)), we have $d_l > d$, hence

$B_l > d$. Note that for the same reason, $1/\kappa_2 > 1/\kappa_2^l$. Because $\phi_{\kappa^l}^a$ is κ^l -homogeneous of degree 1, by (9.6) we thus have

$$1 \geq (B_l + B)\kappa_2^l > (d + B)\kappa_2^l,$$

which implies that

$$B < \frac{1}{\kappa_2^l} - d \leq \frac{1}{\kappa_2} - \frac{1}{\kappa_1 + \kappa_2} = \frac{d}{m} \leq \frac{d}{2}.$$

Let us localize to frequencies of size $\lambda > 1$ by putting

$$\widehat{\nu_k^\lambda}(\xi) := \chi_1\left(\frac{\xi}{\lambda}\right) \widehat{\nu_k}(\xi),$$

where χ_1 is a smooth bump function supported where $|\xi| \sim 1$. We claim that the measures ν_k^λ satisfy the following estimates, uniformly in $k \geq k_0$, provided k_0 is sufficiently large and ε' sufficiently small:

$$(9.8) \quad \|\widehat{\nu_k^\lambda}\|_\infty \leq C\lambda^{-1/B};$$

$$(9.9) \quad \|\nu_k^\lambda\|_\infty \leq C\lambda^{2-1/B}.$$

Indeed,

$$\widehat{\nu_k^\lambda}(\xi) = \chi_1\left(\frac{\xi}{\lambda}\right) \int e^{-i\left[\xi_1 x_1 + \xi_2 \left(2^{(m\kappa_1^l - \kappa_2^l)k} x_2 + x_1^m \omega(2^{-\kappa_1^l k} x_1)\right) + \xi_3 \phi_k(x)\right]} \tilde{\eta}(x) dx,$$

which, in the limit as $k \rightarrow \infty$, simplifies as

$$\widehat{\nu_\infty^\lambda}(\xi) = \chi_1\left(\frac{\xi}{\lambda}\right) \int e^{-i[\xi_1 x_1 + \xi_2 \omega(0) x_1^m + \xi_3 \phi_{\kappa^l}^a(x)]} \tilde{\eta}(x) dx.$$

Now, if $|\xi_3| \geq c|(\xi_1, \xi_2)|$, then an application of van der Corput's lemma to the integration in x_2 yields $|\widehat{\nu_\infty^\lambda}(\xi)| \lesssim |\xi_3|^{-1/B}$ (cf. (9.6)), and if $|\xi_3| \ll |(\xi_1, \xi_2)|$, we may apply van der Corput's lemma to the x_1 -integration and obtain $|\widehat{\nu_\infty^\lambda}(\xi)| \lesssim |(\xi_1, \xi_2)|^{-1/2}$. Since $B \geq 2$, and because van der Corput's estimates are stable under small perturbations, we thus obtain (9.8).

In order to verify (9.9), observe that

$$\nu_\infty^\lambda(x_1, x_2, x_3) = \lambda^3 \int \widehat{\chi_1}(\lambda(x_1 - y_1), \lambda(x_2 - \omega(0)y_1^m), \lambda(x_3 - \phi_{\kappa^l}^a(y_1, y_2))) \tilde{\eta}(y) dy_1 dy_2,$$

hence

$$|\nu_\infty^\lambda(x_1, x_2, x_3)| \leq \lambda^3 \int \rho(\lambda x_1 - \lambda y_1) \rho(\lambda x_3 - \lambda \phi_{\kappa^l}^a(y_1, y_2)) |\tilde{\eta}(y_1, y_2)| dy_1 dy_2,$$

where ρ and η_1 are suitable, non-negative Schwartz functions, and η_1 localizes again to $U(v)$. However, since $|\partial_2^B \phi_{\kappa^l}^a(y_1, y_2)| \simeq 1$ on the domain of integration, classical sublevel estimates, originating in work by van der Corput [5] (see also [1], and [4],[9]), essentially would imply that the integral with respect to y_2 can be estimated by $O(\lambda^{-1/B})$,

uniformly in y_1 and λ (at least, if ρ had compact support). To be more precise, we can argue as follows: By means of Fourier inversion, re-write

$$\begin{aligned} & \int \rho(\lambda x_1 - \lambda y_1) \rho(\lambda x_3 - \lambda \phi_{\kappa^l}^a(y_1, y_2)) |\tilde{\eta}|(y_1, y_2) dy_1 dy_2, \\ &= \int \rho(\lambda x_1 - \lambda y_1) \hat{\rho}(s) e^{is(\lambda x_3 - \lambda \phi_{\kappa^l}^a(y_1, y_2))} |\tilde{\eta}|(y_1, y_2) dy_2 ds dy_1, \end{aligned}$$

and then apply again van der Corput's estimate to the y_2 -integration. This yields

$$\begin{aligned} & \left| \int \rho(\lambda x_1 - \lambda y_1) \rho(\lambda x_3 - \lambda \phi_{\kappa^l}^a(y_1, y_2)) |\tilde{\eta}|(y_1, y_2) dy_1 dy_2 \right|, \\ & \lesssim \int \rho(\lambda x_1 - \lambda y_1) |\hat{\rho}(s)| (1 + \lambda |s|)^{-1/B} |\tilde{\eta}|(y_1, y_2) dy_2 ds dy_1, \end{aligned}$$

which is easily estimated by $C\lambda^{-1-1/B}$, so that we obtain $|\nu_\infty^\lambda(x_1, x_2, x_3)| \leq C\lambda^{2-1/B}$. Observing that our argument is again stable under small perturbations, we thus obtain (9.9).

Interpolating the estimates (9.8) and (9.9), it is again easily seen that we can sum the corresponding estimates over dyadic λ 's and obtain the L^p - L^2 restriction estimate

$$\left(\int |\widehat{f}|^2 d\nu_k \right)^{1/2} \leq C_p \|f\|_{L^p}$$

whenever $p' > 4B$, uniformly in k , for k sufficiently large.

The restriction estimates above are valid in particular for $p' = p'_c$, since, by (9.7), $B < d/2$, so that $p'_c \geq 2d + 2 > 4B$. We have thus again verified (9.4). Q.E.D.

In combination with Proposition 8.3, we immediately obtain

Corollary 9.2. *The restriction estimate in Proposition 4.3 holds true in the Case (c), i.e., when the principal face of the Newton polyhedron of ϕ^a is unbounded.*

Remark 9.3. *When $h_{\text{lin}} \geq 5$, then Case 2 where $\partial_2 \phi_{\kappa^l}^a(v) = 0$ and $\partial_1 \phi_{\kappa^l}^a(v) \neq 0$ could be handled alternatively by means of Drury's Fourier restriction theorem for curves with non-vanishing torsion (cf. Theorem 2 in [7]). This approach will allow to treat the analogous case also for the remaining domain D_{pr} , provided $h_{\text{lin}} \geq 5$, since it does not require the condition $B < d/2$, which may not hold true in D_{pr} ,*

Indeed, if $\partial_1 \phi_{\kappa^l}^a(v) \neq 0$, then $c_1 \neq 0$ in (9.6). Moreover,

$$(9.10) \quad 2 \leq m < a_l = \kappa_2^l / \kappa_1^l < 1 / \kappa_1^l,$$

since $\kappa_1^l A_{l-1} + \kappa_2^l B_{l-1} = 1$ with $B_{l-1} \geq h > 1$, so that $\kappa_2^l < 1$. Observe next that $F : (x_1, c) \mapsto (x_1, c x_1^{a_l})$ provides local smooth coordinates near $v = (1, c_0)$, since the Jacobian J_F of F at the point $(1, c_0)$ is given by $J_F(1, c_0) = 1$. We may therefore fibre the variety S_∞ into the family of curves

$$\gamma_c(x_1) := g_\infty(F(x_1, c)) = (x_1, \omega(0)x_1^m, \phi_{\kappa^l}^a(F(x_1, c))), \quad c \in V(c_0),$$

where $V(c_0)$ is a sufficiently small neighborhood of c_0 , provided ε' is chosen sufficiently small. But, (9.10) implies that the curve $\gamma_{c_0}(x_1) = (x_1, \omega(0)x_1^m, c_1x_1^{1/\kappa_1^l})$ has non-vanishing torsion near v_1 , since $v_1 \neq 0$, and so the same is true for the curves γ_c when c is sufficiently close to c_0 .

If we fibre in a similar way the surface S_k into the family of curves

$$\gamma_c^k(x_1) := g_k(F(x_1, c), \quad c \in V(c_0),$$

then for k sufficiently large and $V(c_0)$ sufficiently small, these curves will have non-vanishing torsion uniformly bounded from above and below, and the measure ν_k will decompose into the direct integral

$$\langle \nu_k, f \rangle = \iint f(\gamma_c^k(x_1)) \tilde{\eta}(x_1, c) dx_1 dc = \int_{V(c_0)} \int_{W(v_1)} f d\Gamma_c dc,$$

where $\tilde{\eta}$ is a smooth function with compact support in $W(v_1) \times V(c_0)$ and $W(v_1)$ a sufficiently small neighborhood of v_1 , where $d\Gamma_c$ is a measure which has a smooth density with respect to the arclength measure on the curve γ_c^k .

We may thus apply Drury's Fourier restriction theorem for curves with non-vanishing torsion (cf. Theorem 2 in [7]) to the measures $d\Gamma_c$ and obtain that

$$\left(\int_{W(v_1)} |\hat{f}|^2 d\Gamma_c \right)^{\frac{1}{2}} \leq C_p \|f\|_{L^p(\mathbb{R}^3)},$$

provided $p' > 7$ and $2 \leq p'/6$, i.e., if $p' \geq 12$. The constant C_p will then be independent of c provided the neighborhoods $V(c_0)$ and $W(v_1)$ are sufficiently small and k is sufficiently large. But, if $h_{\text{lin}} \geq 5$, then we do have $p'_c \geq 2(h_{\text{lin}} + 1) \geq 12$, so that we do obtain estimate (9.4) also in this way.

10. RESTRICTION ESTIMATES IN THE DOMAIN D_{pr} WHEN $h_{\text{lin}}(\phi) \geq 5$

What remains to be studied is the piece of the surface S corresponding to the domain D_{pr} , in the cases (a) and (b). In this domain, the upper bound $B < d/2$ for the multiplicity B of real roots will in general no longer be true, not even the weaker condition $B < h^r(\phi)/2$, which would still suffice for the previous argument, as the following example shows.

Example 10.1.

$$\phi(x_1, x_2) := (x_2 - x_1^2 - x_1^3)(x_2 - x_1^2 - x_1^4)^3.$$

Here, $\phi_{\text{pr}}(x_1, x_2) = (x_2 - x_1^2)^4$, the multiplicity of the root x_1^2 satisfies $4 > d(\phi) = 8/3$, so that the coordinates (x_1, x_2) are not adapted to ϕ . Adapted coordinates are given by $y_1 := x_1$, $y_2 := x_2 - x_1^2$, and in these coordinates ϕ is given by

$$\phi^a(y_1, y_2) = (y_2 - y_1^3)(y_2 - y_1^4)^3.$$

$\mathcal{N}(\phi^a)$ has three vertices $(A_0, B_0) := (0, 4)$, $(A_1, B_1) := (3, 3)$ and $(A_2, B_2) := (0, 15)$, with corresponding edges $\gamma_1 := [(0, 4), (3, 3)]$ and $\gamma_2 := [(3, 3), (0, 15)]$, and associated weights $\kappa^1 := (1/12, 1/4)$ and $\kappa^2 := (1/15, 4/15)$. Moreover, one easily computes by

means of (1.11) that $h_1 = 11/4$ and $h_2 = 13/5$. We thus see that $h^r(\phi) = h_1 = 11/4$. The multiplicity of the root y_1^3 associated to the first edge γ_1 lying above the bi-sectrix is $1 < (8/3)/2$ and thus satisfies the condition (9.7), whereas the root y_1^4 of multiplicity $B = 3$ associated to the edge γ_2 below the bi-sectrix does not even satisfy $B < h^r(\phi)$, since $3 > 11/4$.

The study of the domain D_{pr} will therefore require finer decompositions into further transition and homogeneous domains (with respect to further weights). These will be devised by means of an iteration scheme, resembling somewhat Varchenko's algorithm for the construction of adapted coordinates. Note that the latter algorithm also shows that the principal root jet ψ is actually a polynomial

$$(10.1) \quad \psi(x_1, x_2) = cx_1^m + \cdots + c_{\text{pr}} x_1^a$$

of degree $a = a_{l_{\text{pr}}}$ in the cases (a) and (b) (cf. [12]).

10.1. First step of the algorithm. Let us begin with Case (a), where $D_{\text{pr}} = \{(x_1, x_2) : 0 < x_1 < \delta, |x_2 - \psi(x_1)| \leq Nx_1^a\}$, with a possibly large constant $N > 0$. We then put $D_{(1)} := D_{\text{pr}}$, $\phi^{(1)} := \phi^a$, $\psi^{(1)} := \psi$ and $a_{(1)} := a$, $\kappa^{(1)} := \kappa^{l_{\text{pr}}}$, so that $D_{(1)}$ can be re-written as

$$D_{(1)} = \{(x_1, x_2) : 0 < x_1 < \delta, |x_2 - \psi^{(1)}(x_1)| \leq Nx_1^{a_{(1)}}\}.$$

As in the discussion of the domains D_l in the previous section, we can cover the domain $D_{(1)}$ by finitely many narrow domains of the form

$$D_{(1)}(c_0) := \{(x_1, x_2) : 0 < x_1 < \delta, |x_2 - \psi(x_1) - c_0 x_1^{a_{(1)}}| \leq \varepsilon x_1^{a_{(1)}}\},$$

where $\varepsilon > 0$ can be chosen as small as we need, and where $0 \leq c_0 \leq N$. Fix any of these domains, and put again $v := (1, c_0)$.

We distinguish between the cases where $\partial_2 \phi_{\kappa^{(1)}}^{(1)}(v) \neq 0$ (Case 1), $\partial_2 \phi_{\kappa^{(1)}}^{(1)}(v) = 0$ and $\partial_1 \phi_{\kappa^{(1)}}^{(1)}(v) \neq 0$ (Case 2), and the case where $\nabla \phi_{\kappa^{(1)}}^{(1)}(v) = 0$ (Case 3).

Now, in Case 1, we can argue as in the corresponding case in Section 9, since our arguments in that case did not make use of the condition $l > l_{\text{pr}}$.

In Case 2, the argument given in Section 9 may fail, since it made use of the estimate $B < d/2$, which here no longer may hold true. However, as explained in Remark 9.3, if $h_{\text{lin}} \geq 5$, we may use the alternative argument based on Drury's restriction estimate for curves in this case.

If Case 3 does not appear for any choice of c_0 , then we stop our algorithm and are done.

Otherwise, assume Case 3 applies to c_0 , so that $c_0 x_1^{a_{(1)}}$ is a root of $\phi_{\kappa^{(1)}}^{(1)}$, say of multiplicity $M_1 \geq 2$. In this case, we define new coordinates y by putting

$$(10.2) \quad y_1 := x_1 \quad \text{and} \quad y_2 := x_2 - \psi^{(2)}(x_1),$$

where

$$\psi^{(2)}(x_1) := \psi(x_1) + c_0 x_1^{a_{(1)}}.$$

We denote by $x = s_{(2)}(y)$ the corresponding change of coordinates, which in general is a fractional shear only, since the exponent $a_{(1)} = a$ may be non-integer (but rational). In these coordinates (y_1, y_2) , ϕ is given by $\phi^{(2)} := \phi \circ s_{(2)}$, and the domain $D_{(1)}(c_0)$ becomes the domain

$$D'_{(1)} := \{(y_1, y_2) : 0 < y_1 < \delta, |y_2| \leq \varepsilon y_1^{a_{(1)}}\},$$

which is still $\kappa^{(1)}$ homogeneous.

Let us see to which extent the Newton polyhedra of $\phi^{(1)}$ and $\phi^{(2)}$ will differ.

Claim 1. The Newton polyhedra of $\phi^{(1)}$ and $\phi^{(2)}$ agree in the region above the bi-sectrix. In particular, the line $\Delta^{(m)}$ intersects the boundary of the augmented Newton polyhedron $\mathcal{N}^r(\phi^{(1)}) = \mathcal{N}^r(\phi^a)$ at the same point as the augmented Newton polyhedron $\mathcal{N}^r(\phi^{(2)})$ of $\phi^{(2)}$, so that we can use the modified “adapted” coordinates (10.2) in place of our earlier adapted coordinates to compute the r -height of ϕ .

To see this, observe that $\phi^{(2)}(x_1, x_2) = \phi^{(1)}(x_1, x_2 + c_0 x_1^{a_{(1)}})$, where the exponent $a_{(1)}$ is just the reciprocal of the slope of the line containing the principal face of $\phi^{(1)} = \phi^a$. This implies that the edges of $\mathcal{N}(\phi^{(1)})$ and $\mathcal{N}(\phi^{(2)})$ which lie strictly above the bi-sectrix and do not intersect it are the same (compare corresponding discussions in [12]). Moreover, if $\gamma_{(1)} = [(A_{(0)}, B_{(0)}), (A_{(1)}, B_{(1)})] = [(A_{l_{\text{pr}}-1}, B_{l_{\text{pr}}-1}), (A_{l_{\text{pr}}}, B_{l_{\text{pr}}})]$ is the principal face of $\mathcal{N}(\phi^{(1)})$, then it is easy to see that the principal face of $\mathcal{N}(\phi^{(2)})$ is given by the edge $\gamma'_{(1)} := [(A_{(0)}, B_{(0)}), (A'_{(1)}, B'_{(1)})]$, where

$$(10.3) \quad A'_{(1)} := A_{(1)} + a_{(1)}(B_{(1)} - M_1), \quad B'_{(1)} = M_1,$$

(write $\phi_{\kappa^{(1)}}^{(1)}$ in the normal form (1.10) and use that $c_0 x_1^{a_{(1)}}$ is a root of of multiplicity M_1 of $\phi_{\kappa^{(1)}}^{(1)}$). Observe also that $M_1 \leq h$, because ϕ^a is in adapted coordinates. We thus see that the right endpoint of $\gamma'_{(1)}$ still lies on or below the bi-sectrix. This proves the claim.

Observe that our considerations show that it suffices to study the contributions of narrow domains of the form

$$(10.4) \quad D'_{(1)} = \{(x_1, x_2) : 0 < x_1 < \delta, |x_2 - \psi^{(2)}(x_1)| \leq \varepsilon x_1^{a_{(1)}}\}$$

in place of $D_{(1)}$ (these actually depend on the choice of real root of $\phi_{\kappa^{(1)}}^{(1)}$ - this corresponds to a “fine splitting” of roots of ϕ , in the case where ϕ is analytic).

Case A. $\mathcal{N}(\phi^{(2)}) \subset \{(t_1, t_2) : t_2 \geq B'_{(1)} = M_1\}$. In this case, we again stop our algorithm.

Case B. $\mathcal{N}(\phi^{(2)})$ contains a point below the line where $t_2 = B'_{(1)} = M_1$.

Then $\mathcal{N}(\phi^{(2)})$ will contain a further compact edge

$$\gamma_{(2)} = [(A'_{(1)}, B'_{(1)}), (A_{(2)}, B_{(2)})],$$

so that $(A'_{(1)}, B'_{(1)})$ is a vertex at which the edges $\gamma'_{(1)}$ and $\gamma_{(2)}$ meet. Determine the weight $\kappa^{(2)}$ by requiring that $\gamma_{(2)}$ lies on the line

$$\kappa_1^{(2)} t_1 + \kappa_2^{(2)} t_2 = 1,$$

and put $a_{(2)} := \kappa_2^{(2)} / \kappa_1^{(2)}$. Then clearly $a_{(1)} < a_{(2)}$.

Next, we decompose the domain $D'_{(1)}$ into the domains

$$E_{(1)} := \{(x_1, x_2) : 0 < x_1 < \delta, Nx_1^{a_{(2)}} < |x_2 - \psi^{(2)}(x_1)| \leq \varepsilon x_1^{a_{(1)}}\}$$

and

$$D_{(2)} := \{(x_1, x_2) : 0 < x_1 < \delta, |x_2 - \psi^{(2)}(x_1)| \leq Nx_1^{a_{(2)}}\},$$

where $N > 0$ will be a sufficiently large constant.

The contributions by the transition domain $E_{(1)}$ can be estimated in exactly the same way as we did for the domains E_l in Section 8. Indeed, notice that our arguments for the domains E_l did apply to any $l \geq l_0$ as long as $B_l \geq 1$, so that this statement is immediate when $c_0 = 0$, where the coordinates y in (10.2) do agree with our original adapted coordinates. When $c_0 \neq 0$, there are two minor twists in the arguments needed: firstly, observe that Lemma 8.1 remains valid for $\Phi = \phi^{(2)}$ and the domain

$$E_{(1)}^a := \{(y_1, y_2) : 0 < y_1 < \delta, 2^M y_1^{a_{(2)}} < |y_2| \leq 2^{-M} y_1^{a_{(1)}}\}$$

corresponding to the domain $E_{(1)}$ in the coordinates (10.2) when $\varepsilon = 2^{-M}$ and $N = 2^M$. The fact that $a_{(2)}$ may be non-integer, but rational, say $a_{(2)} = p/q$, with $p, q \in \mathbb{N}$, requires minor changes of the proof only: just consider the Taylor expansion of the smooth function $\Phi(y_1^q, y_2)$. Secondly, if we define in analogy with h_l in (1.11) the corresponding quantity associated to the edges $\gamma'_{(1)}$ and $\gamma_{(2)}$ of $\mathcal{N}(\phi^{(2)})$ by

$$h_{(1)} := \frac{1 + m\kappa_1^{(1)} - \kappa_2^{(1)}}{\kappa_1^{(1)} + \kappa_2^{(1)}} = h_{l_{\text{pr}}} \quad \text{and} \quad h_{(2)} := \frac{1 + m\kappa_1^{(2)} - \kappa_2^{(2)}}{\kappa_1^{(2)} + \kappa_2^{(2)}},$$

then Claim 1 shows that $\max\{h_{(1)}, h_{(2)}\} \leq h^r(\phi)$, which replaces the condition $\max\{h_l, h_{l+1}\} \leq h^r(\phi)$ that was needed in the proof of Proposition 8.3.

10.2. Further steps of the algorithm. We are thus left with the domains $D_{(2)}$, which formally look exactly like $D_{(2)}$, only with $\psi^{(1)}$ replaced by $\psi^{(2)}$ and $a_{(1)}$ replaced by $a_{(2)}$. This allows to iterate this first step of the algorithm which led from $D_{(1)}$ to $D_{(2)}$, producing in this way nested sequences of domains

$$D_{\text{pr}} = D_{(1)} \supset D_{(2)} \supset \cdots \supset D_{(l)} \supset D_{(l+1)} \supset \cdots,$$

of the form

$$D_{(l)} := \{(x_1, x_2) : 0 < x_1 < \delta, |x_2 - \psi^{(l)}(x_1)| \leq Nx_1^{a_{(l)}}\},$$

where the functions $\psi^{(l)}$ are of the form

$$\psi^{(l)}(x_1) = \psi(x_1) + \sum_{j=1}^{l-1} c_{j-1} x_1^{a_{(j)}},$$

with real coefficients c_j , and where the exponents $a_{(j)}$ form a strictly increasing sequence

$$a = a_{(1)} < a_{(2)} < \cdots < a_{(l)} < a_{(l+1)} < \cdots$$

of rational numbers.

Moreover, each of the domains $D_{(l)}$ will be covered by a finite number of domains $D'_{(l)}$ of the form

$$(10.5) \quad D'_{(l)} = \{(x_1, x_2) : 0 < x_1 < \delta, |x_2 - \psi^{(l+1)}(x_1)| \leq \varepsilon x_1^{a_{(l)}}\},$$

where $\varepsilon > 0$ can be chosen as small as we please. These in return will decompose as

$$(10.6) \quad D'_{(l)} = E_{(l)} \cup D_{(l+1)},$$

where $E_{(l)}$ is a transition domain of the form

$$E_{(l)} := \{(x_1, x_2) : 0 < x_1 < \delta, N x_1^{a_{(l+1)}} < |x_2 - \psi^{(l+1)}(x_1)| \leq \varepsilon x_1^{a_{(l)}}\}$$

Putting

$$\phi^{(l)}(x_1, x_2) := \phi(x_1, x_2 + \psi^{(l)}(x_1)),$$

one finds that the Newton polyhedron $\mathcal{N}(\phi^{(l+1)})$ agrees with that one of $\phi^a = \phi^{(1)}$ in the region above the bi-sectrix, and it will have subsequent “edges”

$$\begin{aligned} \gamma'_{(1)} &= [(A_{(0)}, B_{(0)}), (A'_{(1)}, B'_{(1)})], \gamma'_{(2)} = [(A'_{(1)}, B'_{(1)}), (A'_{(2)}, B'_{(2)})], \dots, \\ \gamma'_{(l)} &= [(A'_{(l-1)}, B'_{(l-1)}), (A'_{(l)}, B'_{(l)})], \gamma_{(l+1)} = [(A'_{(l)}, B'_{(l)}), (A_{(l+1)}, B_{(l+1)})], \end{aligned}$$

crossing or lying below the bi-sectrix, at least (possibly more). In fact, it is possible that some of these “edges” degenerate and become a single point (we then shall still speak of an edge, with a slight abuse of notation). The edge with index l will lie on a line

$$L^{(l)} := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1^{(l)} t_1 + \kappa_2^{(l)} t_2 = 1\},$$

where $a_{(l)} = \kappa_2^{(l)} / \kappa_1^{(l)}$. Moreover, $c_{l-1} x_1^{a_{(l)}}$ is any real root of the $\kappa^{(l)}$ -homogeneous polynomial $\phi_{\kappa^{(l)}}^{(l)}$, of multiplicity $M_l \geq 2$. Notice that when ϕ is real-analytic, then this just means that $\psi^{(l)}$ is a leading term of a root of ϕ belonging to the cluster of roots defined by ψ (in the sense of [16]). Our algorithm thus follows any possible “fine splitting” of the roots belonging to this cluster, and the domains $D_{(l)}$ etc. depend on the branches of these roots that we chose along the way.

By our construction, we see that $M_l = B'_{(l)}$, which shows that the sequence of multiplicities is decreasing, i.e.,

$$(10.7) \quad M_1 \geq M_2 \geq \cdots \geq M_l \geq M_{l+1} \geq \cdots$$

Observe also that the transition domains $E_{(l)}$ can be handled by the same reasoning that we had applied to $E_{(1)}$.

When will our algorithm stop? Clearly, this will happen at step l when $\phi_{\kappa(l)}^{(l)}$ has no real root, so that only Case 1 and Case 2 will arise at this step. In that case, we do obtain the desired Fourier restriction estimate for the piece of surface corresponding to $D_{(l)}$, just by the same reasoning that we applied in Section 9. Otherwise, we shall also stop our algorithm in step l when

$$(10.8) \quad \mathcal{N}(\phi^{(l+1)}) \subset \{(t_1, t_2) : t_2 \geq B'_{(l)} = M_l\}.$$

In this situation, the domain which still needs to be understood is the domain $D'_{(l)}$ given by (10.5).

Notice that in this case, Condition (R) implies that there is a function $\tilde{\psi}^{(l+1)} \sim \psi^{(l+1)}$ such that ϕ factors as

$$(10.9) \quad \phi(x_1, x_2) = (x_2 - \tilde{\psi}^{(l+1)}(x_1))^{M_l} \tilde{\phi}(x_1, x_2),$$

where $\tilde{\phi}$ is fractionally smooth. This means that Lemma 8.2 (respectively its immediate extension to fractionally smooth functions) applies to the function $\Phi(y_1, y_2) := \phi(y_1, y_2 + \tilde{\psi}^{(l+1)}(y_1))$, and since the domain $D'_{(l)}$ can be regarded as a generalized transition domain, like the domains $E_{l_{\text{pr}}-1}$ that appeared when the principal face of ϕ^a was an unbounded horizontal face, we can argue in the same way as we did for the domains $E_{l_{\text{pr}}-1}$ in Section 8 to derive the required restriction estimates for the piece of S corresponding to $D'_{(l)}$.

There is finally the possibility that our algorithm does not terminate. In this case, (10.7) shows that the sequence of integers M_l will eventually become constant. We then choose L minimal so that $M_l = M_L$ for all $l \geq L$. Note that, by our construction, $M_L \geq 2$. For every $l \geq L + 1$, the point $(A, B) := (A_{(L)}, B_{(L)}) = (A_L, M_L)$ will be a vertex of $\mathcal{N}(\phi^{(l)})$ which is contained in the line $L^{(l)}$, whose slope $1/a_{(l)}$ tends to zero as $l \rightarrow \infty$, and $\mathcal{N}(\phi^{(l)})$ is contained in the half-plane bounded by $L^{(l)}$ from below.

Notice also that there is a fixed rational number $1/q$, with q integer, such that every $a_{(l)}$ is a multiple of $1/q$. This can be proven in the same way as the corresponding statement in [13] on p. 240.

We can thus apply a classical theorem of E. Borel in a similar way as [12] in order to show that there is a smooth function h of x_1 whose Taylor series expansion is given by the formal series

$$h(x_1) \sim \psi(x_1^q) + \sum_{j=1}^{\infty} c_{j-1} x_1^{qa_{(j)}}.$$

If we put $\psi^{(\infty)}(x_1) := h(x_1^{1/q})$ and set $\phi^{(\infty)}(y_1, y_2) := \phi(y_1, y_2 - \psi^{(\infty)}(y_1))$, it is then easily seen that a straight-forward adaption of the proof Theorem 5.1 in [12] to show that $\mathcal{N}(\phi^{(\infty)}) \subset \{(t_1, t_2) : t_2 \geq B\}$. Therefore, Condition (R) in Theorem 1.7 implies that, possibly after adding a flat function to $\psi^{(\infty)}$, we may assume that ϕ factors as $\phi(x_1, x_2) = (x_2 - \psi^{(\infty)}(x_1))^B \tilde{\phi}(x_1, x_2)$, which means that the analogue of (10.9) holds true. We can thus argue as before to complete also this case, hence also the discussion of the Case (a) where the principal face of $\mathcal{N}(\phi^a)$ is a compact edge.

Finally, in Case (b) where the principal face of $\mathcal{N}(\phi^a)$ is a vertex, we have that $D_{\text{pr}} = \{(x_1, x_2) : |x_2 - \psi(x_1)| \leq \varepsilon x_1^a\}$, which corresponds to the domain $D'_{(1)}$ in the discussion of Case (a). This means that we can just drop the initial step of the algorithm described before, and from then on may proceed as in Case (a).

We have thus established the desired restriction estimates for the piece of the surface S corresponding to the remaining domain D_{pr} , which completes the proof of Theorem 1.7 in the case where $h_{\text{lin}}(\phi) \geq 5$.

What remains open at this stage is the proof of Proposition 4.3 in the case where $2 \leq h_{\text{lin}}(\phi) < 5$. The discussion of this case requires substantially more refined techniques and will be the content of [15].

11. NECESSARY CONDITIONS, AND PROOF OF PROPOSITION 1.9

We now turn to the proof of Theorem 1.8. We shall prove the following, more general result (notice that we are making no assumption on adaptedness of ϕ here):

Proposition 11.1. *Assume that the coordinates $x = (x_1, x_2)$ are linearly adapted to ϕ , and that the restriction estimate (1.1) holds true in a neighborhood of $x^0 = 0$, where $\rho(x^0) \neq 0$. Consider any fractional shear, say on H^+ , given by*

$$y_1 := x_1, \quad y_2 := x_2 - f(x_1),$$

where f is real-valued and fractionally smooth, but not flat. Let $\phi^f(y) = \phi(y_1, y_2 + f(y_1))$ be the function expressing ϕ in the coordinates $y = (y_1, y_2)$. Then necessarily

$$(11.1) \quad p' \geq 2h^f(\phi) + 2.$$

Theorem 1.8 will follow by choosing for f the principal root jet ψ .

Proof. The proof will be based on suitable Knapp-type arguments.

Let us use the same notation for the Newton polyhedron of ϕ^f as we did for ϕ^a in Section 1, i.e., the vertices of the Newton polyhedron $\mathcal{N}(\phi^f)$ will be denoted by (A_l, B_l) , $l = 0, \dots, n$, where we assume that they are ordered so that $A_{l-1} < A_l$, $l = 1, \dots, n$, with associated compact edges given by the intervals $\gamma_l := [(A_{l-1}, B_{l-1}), (A_l, B_l)]$, $l = 1, \dots, n$, contained in the L_l and associated with the weights κ^l . The unbounded horizontal edge with left endpoint (A_n, B_n) will be denoted by γ_{n+1} . For $l = n+1$, we have $\kappa_1^{n+1} := 0, \kappa_2^{n+1} = 1/B_n$. Again, we put $a_l := \kappa_2^l / \kappa_1^l$, and $a_{n+1} := \infty$.

Because of (1.12), we have to prove the following estimates:

$$(11.2) \quad p' \geq 2d^f + 2;$$

$$(11.3) \quad p' \geq 2h_l^f + 2 \quad \text{for every } l \text{ such that } a_l > m_0.$$

where, according to (1.13),

$$h_l^f = \frac{1 + m_0 \kappa_1^l - \kappa_2^l}{\kappa_1^l + \kappa_2^l}.$$

Let first γ_l be any non-horizontal edge of $\mathcal{N}(\phi^f)$ with $a_l > m_0$, and consider the region

$$D_\varepsilon^a := \{y \in \mathbb{R}^2 : |y_1| \leq \varepsilon^{\kappa_1^l}, |y_2| \leq \varepsilon^{\kappa_2^l}\}, \quad \varepsilon > 0,$$

in the coordinates y . In the original coordinates x , it corresponds to

$$D_\varepsilon := \{x \in \mathbb{R}^2 : |x_1| \leq \varepsilon^{\kappa_1^l}, |x_2 - f(x_1)| \leq \varepsilon^{\kappa_2^l}\}.$$

Assume that ε is sufficiently small. Since

$$\phi^f(\varepsilon^{\kappa_1^l} y_1, \varepsilon^{\kappa_2^l} y_2) = \varepsilon(\phi_{\kappa^l}^f(y_1, y_2) + O(\varepsilon^\delta)),$$

for some $\delta > 0$, where $\phi_{\kappa^l}^f$ denotes the κ^l -principal part of ϕ^f , we have that $|\phi^f(y)| \leq C\varepsilon$ for every $y \in D_\varepsilon^a$, i.e.

$$(11.4) \quad |\phi(x)| \leq C\varepsilon \quad \text{for every } x \in D_\varepsilon.$$

Moreover, for $x \in D_\varepsilon$, because $|f(x_1)| \lesssim |x_1|^{m_0}$ and $m_0 \leq a_l = \kappa_2^l / \kappa_1^l$, we have

$$|x_2| \leq \varepsilon^{\kappa_2^l} + |f(x_1)| \lesssim \varepsilon^{\kappa_2^l} + \varepsilon^{m_0 \kappa_1^l} \lesssim \varepsilon^{m_0 \kappa_1^l}.$$

We may thus assume that D_ε is contained in the box where $|x_1| \leq 2\varepsilon^{\kappa_1^l}$, $|x_2| \leq 2\varepsilon^{m_0 \kappa_1^l}$. Choose a Schwartz function φ_ε such that

$$\widehat{\varphi_\varepsilon}(x_1, x_2, x_3) = \chi_0\left(\frac{x_1}{\varepsilon^{\kappa_1^l}}\right) \chi_0\left(\frac{x_2}{\varepsilon^{m_0 \kappa_1^l}}\right) \chi_0\left(\frac{x_3}{C\varepsilon}\right),$$

where χ_0 is again a smooth cut-off function supported in $[-2, 2]$ identically 1 on $[-1, 1]$.

Then by (11.4) we see that $\widehat{\varphi_\varepsilon}(x_1, x_2, \phi(x_1, x_2)) \geq 1$ on D_ε , hence, if $\rho(0) \neq 0$, then

$$\left(\int_S |\widehat{\varphi_\varepsilon}|^2 \rho d\sigma\right)^{1/2} \geq C_1 |D_\varepsilon|^{1/2} = C_1 \varepsilon^{(\kappa_1^l + \kappa_2^l)/2},$$

where $C_1 > 0$ is a positive constant. Since $\|\varphi_\varepsilon\|_p \simeq \varepsilon^{((1+m_0)\kappa_1^l+1)/p'}$, we find that the restriction estimate (1.1) can only hold if

$$p' \geq 2 \frac{(1+m_0)\kappa_1^l + 1}{\kappa_1^l + \kappa_2^l} = 2h_l^f + 2.$$

The case $l = n + 1$, where γ_l is the horizontal edge, so that $h_l^f = B_n - 1$, requires a minor modification of this argument. Observe that, by Taylor expansion, in this case ϕ^f can be written as

$$(11.5) \quad \phi^f(y_1, y_2) = y_2^{B_n} h(y_1, y_2) + \sum_{j=0}^{B_n-1} y_2^j g_j(y_1),$$

where the functions g_j are flat and h is fractionally smooth and continuous at the origin. Choose $\delta > 0$, and define here

$$D_\varepsilon^a := \{y \in \mathbb{R}^2 : |y_1| \leq \varepsilon^\delta, |y_2| \leq \varepsilon^{\kappa_2^l}\}, \quad \varepsilon > 0.$$

Then (11.5) shows that again $|\phi^f(y)| \leq C\varepsilon$ for every $y \in D_\varepsilon^a$, so that (11.4) holds true again. Moreover, for $x \in D_\varepsilon$, we now find that

$$|x_2| \leq \varepsilon^{\kappa_2^l} + |f(x_1)| \lesssim \varepsilon^{\kappa_2^l} + \varepsilon^{m_0\delta} \lesssim \varepsilon^{m_0\delta}$$

for δ sufficiently small. Choosing

$$\widehat{\varphi_\varepsilon}(x_1, x_2, x_3) = \chi_0\left(\frac{x_1}{\varepsilon^\delta}\right) \chi_0\left(\frac{x_2}{\varepsilon^{m_0\delta}}\right) \chi_0\left(\frac{x_3}{C\varepsilon}\right),$$

arguing as before we find that here (1.1) implies that

$$p' \geq 2 \frac{(1 + m_0)\delta + 1}{\delta + \kappa_2^l} \quad \text{for every } \delta > 0,$$

hence $p' \geq 2B_n = 2h_{l_{n+1}}^f + 2$. This finishes the proof of (11.3).

Notice finally that the argument for the non-horizontal edges still works if we replace the line L_l by the line L^f and the weight κ^l by the weight κ^f associated with that line. Since here $m_0\kappa_1^f = \kappa_2^f$, this leads to the condition (11.2). Q.E.D.

Proposition 11.1 also allows to give a short, but admittedly indirect proof of Proposition 1.9, which will make use of Theorem 1.4, too.

Proof of Proposition 1.9. Recall that we assume that the original coordinates (x_1, x_2) are linearly adapted to ϕ .

In order to prove (a), assume furthermore that the coordinates (x_1, x_2) are not adapted to ϕ , and let $f(x_1)$ be any non-flat fractionally smooth, real function $f(x_1)$, with corresponding fractional shear, say in H^+ . We have to show that

$$(11.6) \quad h^f(\phi) \leq h^r(\phi).$$

We begin with the special case where ϕ is analytic, then Theorem 1.4 shows that the restriction estimate (1.1) holds true for $p = p_c$, where $p'_c = 2h^r(\phi) + 2$. Moreover, choosing ρ so that $\rho(x^0) \neq 0$, then Proposition 11.1 implies that $p' \geq 2h^f(\phi) + 2$. Combining these estimates we obtain (11.6).

The case of a general smooth, finite type ϕ can be reduced to the previous case. To this end, denote by ϕ_N the Taylor polynomial of degree N centered at the origin. It is not difficult to show that if N is sufficiently large, then

$$h^r(\phi) = h^r(\phi_N) \quad \text{and} \quad h^f(\phi_N) = h^f(\phi).$$

Since (11.6) holds true for ϕ_N , we thus see that it holds true also for ϕ .

In order to prove (b), we assume that the coordinates (x_1, x_2) are adapted to ϕ , so that $d(\phi) = h(\phi)$. We have to prove that

$$(11.7) \quad \tilde{h}^r(\phi) = d(\phi).$$

Let us first observe that Theorem 1.1 and Proposition 11.1 imply, in a similar way as in the proof of (a), that $2h(\phi) + 2 \geq 2h^f(\phi) + 2$, hence $d(\phi) \geq h^f(\phi)$. We thus see that

$$\tilde{h}^r(\phi) \leq d(\phi).$$

On the other hand, when the principal face $\pi(\phi)$ is compact, then we can choose a support line

$$L = \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1 t_1 + \kappa_2 t_2 = 1\}$$

to the Newton polyhedron of ϕ containing $\pi(\phi)$ and such that $0 < \kappa_1 \leq \kappa_2$. We then put $f(x_1) := x_1^{m_0}$, where $m_0 := \kappa_2/\kappa_1$. Then $d(\phi) = 1/(\kappa_1 + \kappa_2) = d^f \leq h^f(\phi) \leq \tilde{h}^r(\phi)$, and we obtain (11.7).

Assume finally that $\pi(\phi)$ is an unbounded horizontal half-line, with left endpoint (A, B) , where $A < B$. We then choose $f_n(x_1) := x_1^n$, $n \in \mathbb{N}$. Then it is easy to see that for n sufficiently large, the line L^{f_n} will pass through the point (A, B) , and thus $\lim_{n \rightarrow \infty} h^{f_n}(\phi) = B = d(\phi)$. Therefore, $\tilde{h}^r(\phi) \geq d(\phi)$, which shows that (11.7) is valid also in this case. Q.E.D.

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